

POLYTOPES WITH MASS LINEAR FUNCTIONS, PART I

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ABSTRACT. Let Δ be a n -dimensional polytope that is simple, that is, exactly n facets meet at each vertex. An affine function is “mass linear” on Δ if its value on the center of mass of Δ depends linearly on the positions of the supporting hyperplanes. On the one hand, we show that certain types of symmetries of Δ give rise to nonconstant mass linear functions on Δ . On the other hand, we show that most polytopes do not admit any nonconstant mass linear functions. Further, if every affine function is mass linear on Δ , then Δ is a product of simplices. Our main result is a classification of all smooth polytopes of dimension ≤ 3 which admit nonconstant mass linear functions. In particular there is only one family of smooth 3-dimensional polytopes – and no polygons – that admit “essential mass linear functions”, that is, mass linear functions that do not arise from the symmetries described above. In part II, we will complete this classification in the 4-dimensional case.

These results have geometric implications. Fix a symplectic toric manifold (M, ω, T, Φ) with moment polytope $\Delta = \Phi(M)$. Let $\text{Symp}_0(M, \omega)$ denote the identity component of the group of symplectomorphisms of (M, ω) . Any linear function H on Δ generates a Hamiltonian \mathbb{R} action on M whose closure is a subtorus T_H of T . We show that if the map $\pi_1(T_H) \rightarrow \pi_1(\text{Symp}_0(M, \omega))$ has finite image, then H is mass linear. Combining this fact and the claims described above, we prove that in most cases the induced map $\pi_1(T) \rightarrow \pi_1(\text{Symp}_0(M, \omega))$ is an injection. Moreover, the map does not have finite image unless M is a product of projective spaces. Note also that there is a natural maximal compact connected subgroup $\text{Isom}_0(M) \subset \text{Symp}_0(M, \omega)$; there is a natural compatible complex structure J on M , and $\text{Isom}_0(M)$ is the identity component of the group of symplectomorphisms that also preserve this structure. We prove that if the polytope Δ supports no essential mass linear functions, then the induced map $\pi_1(\text{Isom}_0(M)) \rightarrow \pi_1(\text{Symp}_0(M, \omega))$ is injective. Therefore, this map is injective for all 4-dimensional symplectic toric manifolds and is injective in the 6-dimensional case unless M is a $\mathbb{C}P^2$ bundle over $\mathbb{C}P^1$.

with Appendix written with V. Timorin

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Date: June 25, 2008, revised August 10, 2009.

2000 Mathematics Subject Classification. 14M25, 52B20, 53D99, 57S05.

Key words and phrases. simple polytope, Delzant polytope, center of gravity, toric symplectic manifold, symplectomorphism group.

First author partially supported by NSF grant DMS 0604769, and second by NSF grant DMS 0707122.

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1. INTRODUCTION

To begin, we shall give some basic definitions; we then state the main results. These are elaborated in §1.1; their geometric implications are described in §1.2.

Let \mathfrak{t} be a real vector space, let \mathfrak{t}^* denote the dual space, and let $\langle \cdot, \cdot \rangle : \mathfrak{t} \times \mathfrak{t}^* \rightarrow \mathbb{R}$ denote the natural pairing. Let A be an affine space modeled on \mathfrak{t}^* . A **(convex) polytope** $\Delta \subset A$ is the bounded intersection of a finite set of affine half-spaces. Hence, Δ can be written

$$(1.1) \quad \Delta = \bigcap_{i=1}^N \{x \in A \mid h_i(x) \leq \kappa_i\},$$

where the h_i are affine functions on A and the **support numbers** κ_i lie in \mathbb{R} for all $1 \leq i \leq N$; the **outward conormals** are the unique¹ vectors $\eta_i \in \mathfrak{t}$ so that $h_i(y) - h_i(x) = \langle \eta_i, y - x \rangle$ for all x and y in A . When the $\{h_i\}$ are fixed and the dependence on $\{\kappa_i\}$ is important we write $\Delta := \Delta(\kappa)$.

In this paper, we will always assume that A is the affine span of Δ and that the affine span of each facet $F_i := \Delta \cap \{x \in A \mid h_i(x) = \kappa_i\}$ is a hyperplane. In particular, each F_i is nonempty. Moreover, the polytopes we consider are **simple**, that is, $\dim \mathfrak{t}$ facets meet at every vertex.

¹ The ambiguity in choosing h_i is discussed in Remarks 1.8 and 1.13.

We sometimes need stronger assumptions: Given an integer lattice $\ell \subset \mathfrak{t}$, a polytope $\Delta \subset A$ is **rational** if we can choose each outward conormal to lie in ℓ . A rational polytope is **smooth** (or Delzant) if, for each vertex of Δ , the primitive outward conormals to the facets which meet at that vertex form a basis for ℓ . Here, a vector $\eta \in \ell$ is **primitive** if it is not a positive integer multiple of any other lattice element.

The **chamber** \mathcal{C}_Δ of $\Delta := \Delta(\kappa)$ is the connected component that contains κ of the set of all $\kappa' \in \mathbb{R}^N$ such that $\Delta(\kappa')$ is simple. Note that \mathcal{C}_Δ is open. Moreover, for every $\kappa' \in \mathcal{C}_\Delta$ the polytope $\Delta(\kappa')$ is **analogous**² to Δ , that is, it has the property that the intersection $\bigcap_{i \in I} F'_i$ is empty exactly if the intersection $\bigcap_{i \in I} F_i$ is empty for all $I \subset \{1, \dots, N\}$. Since Δ is simple, \mathcal{C}_Δ is a nonempty connected open set.

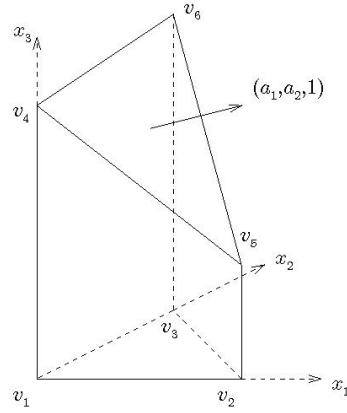


FIGURE 1.1. The polytope Y_a constructed in Example 1.1.

Example 1.1. Let $\{e_i\}_{i=1}^n$ denote the standard basis for \mathbb{R}^n . Given $a = (a_1, a_2) \in \mathbb{R}^2$, define

$$\eta_1 = -e_1, \eta_2 = -e_2, \eta_3 = e_1 + e_2, \eta_4 = -e_3, \eta_5 = e_3 + a_1 e_1 + a_2 e_2, \text{ and}$$

$$\mathcal{C}_a = \left\{ \kappa \in \mathbb{R}^5 \mid \sum_{i=1}^3 \kappa_i > 0 \text{ and } \kappa_4 + \kappa_5 > -a_1 \kappa_1 - a_2 \kappa_2 + \max(0, a_1, a_2) \sum_{i=1}^3 \kappa_i \right\}.$$

Given $\kappa \in \mathcal{C}_a$, let

$$Y = Y_a(\kappa) = \bigcap_{i=1}^5 \{x \in (\mathbb{R}^3)^* \mid \langle \eta_i, x \rangle \leq \kappa_i\};$$

it is illustrated in Figure 1.1. One can simplify the formulas here by translating Y so that its first vertex is at the origin. Then $\kappa_1 = \kappa_2 = \kappa_4 = 0$, and we see that $Y_a(\kappa)$ really

² Two arbitrary polytopes are **analogous** if there is combinatorial equivalence (cf. Definition 1.7) between them in which corresponding facets are parallel. In the case at hand, these conditions reduce to the ones mentioned above.

depends only on two parameters, κ_3 and κ_5 . The vertices now are

$$(0, 0, 0), (\kappa_3, 0, 0), (0, \kappa_3, 0), (0, 0, \kappa_5), (\kappa_3, 0, \kappa_5 - a_1\kappa_3), (0, \kappa_3, \kappa_5 - a_2\kappa_3).$$

The polytope $Y_a(\kappa)$ is simple; it is smooth exactly if $a \in \mathbb{Z}^2$; and its chamber is \mathcal{C}_a .

Let $c_\Delta: \mathcal{C}_\Delta \rightarrow A$ denote the **center of mass** function, which assigns to $\kappa := (\kappa_1, \dots, \kappa_N) \in \mathcal{C}_\Delta$ the center of mass of $\Delta(\kappa)$ (with respect to any affine identification of A with Euclidean space).

Definition 1.2. Let $\Delta \subset A$ be a simple polytope. Let $c_\Delta: \mathcal{C}_\Delta \rightarrow A$ denote the center of mass function. An affine function $H: A \rightarrow \mathbb{R}$ is called **mass linear** with respect to Δ if the composite

$$H \circ c_\Delta: \mathcal{C}_\Delta \rightarrow \mathbb{R}$$

is linear, that is,

$$H(c_\Delta(\kappa)) = \sum_{i=1}^N \gamma_i \kappa_i + C \quad \forall \kappa \in \mathcal{C}_\Delta,$$

where $C \in \mathbb{R}$ and $\gamma_i \in \mathbb{R}$ for all $1 \leq i \leq N$. For short, we shall say that H is a **mass linear function** on Δ ; we sometimes also call (Δ, H) a **mass linear pair**.

We call γ_i the **coefficient** of the support number κ_i (in $H \circ c_\Delta$). If, additionally, $\gamma_i \in \mathbb{Z}$ for all $1 \leq i \leq N$, then we say that H is a **mass linear function with integer coefficients** on Δ .

In practice, we usually choose an identification of A with the linear space \mathfrak{t}^* and consider those H which are elements of the dual space \mathfrak{t} ; cf. Remarks 1.6 and 1.8.

Our primary goal in this paper is to analyze mass linear functions on simple polytopes. On the one hand, we show that certain types of symmetries of Δ give rise to nonconstant functions that are mass linear on Δ . We call nonconstant mass linear functions that arise in this way **inessential** (see Definition 1.14); the others are called **essential**. On the other hand, we show that most polytopes do not admit any nonconstant mass linear functions. Further, if every affine function is mass linear on Δ , then Δ is affine equivalent to a product of simplices. Our main result is a classification of all 2-dimensional simple polytopes and 3-dimensional smooth polytopes which admit nonconstant mass linear functions. In particular, the polytopes constructed in Example 1.1 are the only smooth polytopes of dimension ≤ 3 which admit essential mass linear functions. In part II, we will complete this classification in the 4-dimensional case.

Our first result holds both for an arbitrary polygon and for a smooth polygon, though, as we explain in Remark 1.6 below, the words used have different interpretations in the two cases. This remark also gives the definition of affine equivalence.

Theorem 1.3. *A simple 2-dimensional polytope supports a nonconstant mass linear function exactly if it is a triangle, a trapezoid or a parallelogram. All such functions are inessential. The same statement holds in the smooth case.*

Theorem 1.4. *Let Δ be a smooth 3-dimensional polytope. If there exists an essential mass linear function on Δ , then Δ is affine equivalent to the polytope $Y_a(\kappa)$ constructed in*

Example 1.1 for some $a \in \mathbb{Z}^2$ and $\kappa \in \mathcal{C}_Y$. Further, for any $a \in \mathbb{Z}^2$ there exists an essential mass linear function on the polytope Y_a exactly if $a_1 \neq 0, a_2 \neq 0$ and $a_1 \neq a_2$.

We prove the second statement in Theorem 1.4 by a direct calculation; see Proposition 4.6. However the first claim relies on many of our other results.

These results have geometric implications. Fix a symplectic toric manifold (M, ω, T, Φ) with moment polytope $\Delta = \Phi(M)$. (See §1.2.) Let $\text{Symp}_0(M, \omega)$ denote the identity component of the group of symplectomorphisms of (M, ω) . Any linear function H on Δ generates a Hamiltonian \mathbb{R} action on M whose closure is a subtorus T_H of T . We show that if the map $\pi_1(T_H) \rightarrow \pi_1(\text{Symp}_0(M, \omega))$ has finite image, then H is mass linear. Since (as we explained above) most polytopes do not admit mass linear functions, this implies that in most cases the induced map $\pi_1(T) \rightarrow \pi_1(\text{Symp}_0(M, \omega))$ is an injection. Moreover, since products of simplices are the only polytopes for which every affine function is mass linear, the map does not have finite image unless M is a product of projective spaces.

Note also that there is a natural maximal compact connected subgroup $\text{Isom}_0(M) \subset \text{Symp}_0(M, \omega)$; there is a natural compatible complex structure J on M and hence an associated metric $g_J := \omega(\cdot, J\cdot)$, and we define $\text{Isom}_0(M)$ to be the identity component of the group of isometries of the Riemannian manifold (M, g_J) . It is well known that $\text{Isom}_0(M) \subset \text{Symp}_0(M, \omega)$. Theorem 1.25 below states that if Δ admits no essential mass linear functions, then the natural map from $\pi_1(\text{Isom}_0(M))$ to $\pi_1(\text{Symp}_0(M, \omega))$ is an injection. Since the polytopes constructed in Example 1.1 are exactly the moment polytopes of \mathbb{CP}^2 bundles over \mathbb{CP}^1 (see Example 5.3), Theorems 1.3 and 1.4 have the following consequence.

Theorem 1.5. *Let (M, ω, T, Φ) be a symplectic toric manifold, and let $\text{Isom}_0(M)$ be the identity component of the associated Kähler isometry group. If M is 4-dimensional, or if M is 6-dimensional but is not a \mathbb{CP}^2 bundle over \mathbb{CP}^1 , then the natural map from $\pi_1(\text{Isom}_0(M))$ to $\pi_1(\text{Symp}_0(M, \omega))$ is an injection.*

We end this subsection with some important technical remarks.

Remark 1.6. The group of affine transformations of A acts on simple polytopes $\Delta \subset A$ and affine functions $H: A \rightarrow \mathbb{R}$ by $a \cdot (\Delta, H) = (a(\Delta), H \circ a^{-1})$. In this case, we will say that (Δ, H) and $(a(\Delta), H \circ a^{-1})$ are **affine equivalent**. When we restrict to smooth polytopes we will only allow affine transformations $a: A \rightarrow A$ such that the **associated linear transformation** $\hat{a}: \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ induces an automorphism of the integer lattice; here, \hat{a} is the unique linear map so that $\hat{a}(x - y) = a(x) - a(y)$ for all x and y in \mathfrak{t}^* .

Except for the concepts of “volume” and “moment” considered in §2.2 below, all the notions in this paper are affine invariant. For example, since $a(c_\Delta) = c_{a(\Delta)}$, an affine function H is mass linear on Δ exactly if $H \circ a^{-1}$ is mass linear on $a(\Delta)$. Therefore, by identifying A with \mathfrak{t}^* , we may restrict our attention to polytopes $\Delta \subset \mathfrak{t}^*$. Given two affine equivalent polytopes Δ and Δ' , we will usually simply say that they are the same and write $\Delta = \Delta'$. For example, whether we work with simple or smooth polytopes, there is a unique equivalence class of k -dimensional polytopes with $k + 1$ facets, namely the k -simplex Δ_k . (See Example 1.15.) Hence we say that every such polytope is Δ_k . These conventions mean

that, for example, the interpretation of the word “is” in Theorem 1.3 depends on whether we are considering smooth polytopes or simple ones.

There is another much weaker notion that is sometimes useful.

Definition 1.7. *Two polytopes Δ and Δ' are said to be **combinatorially equivalent** if there exists a bijection of facets $F_i \leftrightarrow F'_i$ so that $\bigcap_{i \in I} F_i \neq \emptyset$ exactly if $\bigcap_{i \in I} F'_i \neq \emptyset$ for all $I \subset \{1, \dots, N\}$.*

We will never use this equivalence relation without explicitly saying so. Note that corresponding facets of combinatorially equivalent polytopes need not be parallel. For example, any two polygons with the same number of edges are combinatorially equivalent.

Remark 1.8. Each outward conormal of a simple polytope $\Delta \subset A$ is only well defined up to multiplication by a positive constant. (See Remark 1.13 for an important normalization convention.) The chamber \mathcal{C}_Δ of Δ depends on these choices but the notion of mass linearity does not – it is well defined. On the other hand, if H is mass linear the coefficients of the support numbers depend on the choice of the outward conormals.

Even once these outward conormals are determined, the affine functions h_i of Equation (1.1) are only well defined up to addition by a constant. However, these choices have no significant consequence. In particular, if $\Delta \subset \mathfrak{t}^*$ we may assume that the h_i are homogeneous. Similarly, we may restrict our attention to homogeneous linear functions $H \in \mathfrak{t}$.

1.1. Additional results on polytopes. Let us begin with a few definitions.

Definition 1.9. *Fix an affine function $H: A \rightarrow \mathbb{R}$ and a simple polytope $\Delta \subset A$. We say that the facet F is **symmetric** (or **H-symmetric**)³ if $\langle H, c_\Delta \rangle: \mathcal{C}_\Delta \rightarrow \mathbb{R}$ does not depend on the support number of F . Otherwise we say that F is **asymmetric** (or **H-asymmetric**). More generally, we say that a face f is **symmetric** if it is the intersection of symmetric facets.*

Definition 1.10. *We say that a facet F of a simple polytope $\Delta \subset A$ is **pervasive** if it meets (i.e. has nonempty intersection with) every other facet of Δ . We say that F is **flat** if there is a hyperplane in \mathfrak{t} that contains the conormal of every facet (other than F itself) that meets F .*

For example, in the polytopes constructed in Example 1.1, the triangular facets are flat but the quadrilaterals are not; the pervasive facets are the quadrilaterals.

In §2, we analyze the key features of symmetric and asymmetric facets. On the one hand, we prove that if H is mass linear on Δ , then H is also mass linear on each symmetric face f of Δ . This allows us to analyze mass linear functions on polytopes which have symmetric faces by “induction” on the dimension of the polytope. On the other hand, we prove that if H is a mass linear function on Δ , then asymmetric facets are very special; they must be either pervasive or flat. As we show in §2.4, this implies that most polytopes do not support any nonconstant mass linear functions.

³When there is no danger of confusion, we will omit the H .

Theorem 1.11. *If $\Delta \subset A$ is a simple polytope that contains no pervasive facets and no flat facets then every mass linear function on Δ is constant.*

On the other hand, there are some polytopes with nonconstant mass linear functions. The simplest type of such functions – inessential functions – arise from symmetries of the polytope.

Definition 1.12. *Let $\Delta \subset A$ be a simple polytope. A **symmetry** of Δ is an affine map $a: A \rightarrow A$ so that $a(\Delta) = \Delta$. When Δ is a smooth polytope we will only allow affine transformations $a: A \rightarrow A$ such that the associated linear transformation induces an automorphism of the integer lattice; cf. Remark 1.6. A **robust symmetry** of Δ is a linear map $\hat{a}: \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ so that for all $\kappa \in \mathcal{C}_\Delta$ there exists an affine map $a_\kappa: A \rightarrow A$ such that*

- (1) a_κ is a symmetry of $\Delta(\kappa)$, and
- (2) \hat{a} is the linear map associated to a_κ .

Finally, we say that two facets F_i and F_j are **equivalent**, denoted $F_i \sim F_j$, if there exists a robust symmetry \hat{a} so that each a_κ takes $F_i(\kappa)$ to $F_j(\kappa)$

It is clear that this defines an equivalence relationship because the robust symmetries form a subgroup of the group of linear transformations. As an example, consider the square

$$\Delta = \{x \in (\mathbb{R}^2)^* \mid -1 \leq x_1 \leq 1 \text{ and } -1 \leq x_2 \leq 1\}.$$

The map $(x_1, x_2) \mapsto (x_2, -x_1)$ is a symmetry of Δ which is not robust. In contrast, the map $(x_1, x_2) \mapsto (-x_1, -x_2)$ is a robust symmetry of Δ . Therefore, each edge is equivalent to the opposite edge.

In §3.1 we give several easily verified criteria for facets to be equivalent. In Proposition 5.5, we show that in the smooth case equivalent facets can also be characterized in terms of the symmetries of the associated toric manifold M . Moreover, in this case, $F_i \sim F_j$ exactly if the corresponding divisors in M are homologous; cf. Remark 5.8.

Remark 1.13. Let $\Delta \subset A$ be a simple polytope. We can (and always will) choose the outward conormals so that for every every robust symmetry $\hat{a}: \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ of Δ and every facet F_i , there exists a facet F_j so that $\hat{a}^*(\eta_i) = \eta_j$. Since the robust symmetries form a group, the fact that this is possible follows easily from Lemma 3.1. If $\Delta \subset \mathfrak{t}^*$ is a smooth polytope, then by definition \hat{a}^* takes each primitive outward conormal to another primitive outward conormal. Hence, when we restrict to smooth polytopes we can (and will) always choose the unique primitive outward conormal to each facet.

Definition 1.14. *Let $\Delta = \bigcap_{i=1}^N \{x \in \mathfrak{t}^* \mid \langle \eta_i, x \rangle \leq \kappa_i\}$ be a simple polytope; let \mathcal{I} denote the set of equivalence classes of facets. We say that $H \in \mathfrak{t}$ is **inessential** iff*

$$(1.2) \quad H = \sum \beta_i \eta_i, \quad \text{where } \beta_i \in \mathbb{R} \ \forall i \quad \text{and} \quad \sum_{i \in I} \beta_i = 0 \quad \forall I \in \mathcal{I}.$$

Otherwise, we say that H is **essential**.

If each equivalence class in \mathcal{I} has just one element then every inessential function on Δ vanishes. More generally, the set of inessential functions is an $(N - |\mathcal{I}|)$ -dimensional subspace of \mathfrak{t} ; see Lemma 3.9.

Example 1.15. As an example, (here and subsequently) let Δ_k denote the standard k -simplex

$$\Delta_k = \{x \in (\mathbb{R}^k)^* \mid x_i \geq 0 \ \forall i \text{ and } \sum x_i \leq 1\};$$

the outward conormals are $\{\eta_i\}_{i=1}^{k+1}$, where $\eta_i = -e_i$ for all $i \in \{1, \dots, k\}$ and $\eta_{k+1} = \sum_{i=1}^k e_i$. The linear map $(x_1, \dots, x_k) \mapsto (-\sum x_i, x_1, \dots, x_{k-1})$ is a robust symmetry of Δ whose transpose permutes the conormals η_i cyclically. Hence, all $k+1$ facets of Δ_k are equivalent, and every $H \in \mathbb{R}^k$ is inessential. For example, $e_j = -\eta_j + \frac{1}{k+1} \sum_{i=1}^{k+1} \eta_i$ for all $j \leq k$. Moreover, a direct computation shows that

$$\langle e_j, c_{\Delta_k}(\kappa) \rangle = -\kappa_j + \frac{1}{k+1} \sum_{i=1}^{k+1} \kappa_i.$$

Example 1.16. Consider a product of simplices $\Delta = \Delta_k \times \Delta_{k'} \subset (\mathbb{R}^k)^* \times (\mathbb{R}^{k'})^* = (\mathbb{R}^{k+k'})^*$. This polytope has $k + k' + 2$ facets. The first $k+1$ facets are of the form $F_i \times \Delta_{k'}$, and are all equivalent to each other. The remaining $k'+1$ facets are also equivalent, and are of the form $\Delta_k \times F_i$. Again, every $H \in \mathbb{R}^{k+k'}$ is inessential and mass linear;

$$\langle e_j, c_{\Delta}(\kappa) \rangle = \begin{cases} -\kappa_j + \frac{1}{k+1} \sum_{i=1}^{k+1} \kappa_i & \text{if } j \leq k \\ -\kappa_{j+1} + \frac{1}{k'+1} \sum_{i=1}^{k'+1} \kappa_{k+1+i} & \text{otherwise} \end{cases}$$

A nearly identical remark applies if Δ is the product of three or more simplices.

Here, the **product** of simple polytopes $\hat{\Delta} \subset \hat{A}$ and $\tilde{\Delta} \subset \tilde{A}$ is $\Delta = \hat{\Delta} \times \tilde{\Delta} \subset \hat{A} \times \tilde{A}$. This is a simple polytope; it is smooth exactly if $\hat{\Delta}$ and $\tilde{\Delta}$ are smooth.

Example 1.17. The triangular facets of the polytope $Y = Y_a(\kappa)$ defined in Example 1.1 are equivalent because they are exchanged by the affine reflection $a_\kappa: (\mathbb{R}^3)^* \rightarrow (\mathbb{R}^3)^*$ defined by

$$a_\kappa(x_1, x_2, x_3) = (x_1, x_2, -a_1x_1 - a_2x_2 - x_3 + \kappa_5 - \kappa_4).$$

Hence, $H = \eta_4 - \eta_5 = -a_1e_1 - a_2e_2 - 2e_3$ is inessential.

Although the definition of an inessential function looks somewhat opaque at first glance, this is one of our most important notions. As we shall see in Corollary 1.28, in the smooth case inessential functions have a very natural interpretation in terms of the geometry of the associated toric manifold. Formula (1.2) is also closely tied to algebraic properties of robust symmetries; cf. Remark 3.6.

We now show that every inessential function is mass linear; cf. Example 1.15.

Proposition 1.18. *Let $\Delta \subset \mathfrak{t}^*$ be a simple polytope; let \mathcal{I} denote the set of equivalence classes of facets. If $H \in \mathfrak{t}$ is inessential, write $H = \sum \beta_i \eta_i$, where the β_i satisfy the conditions of Equation (1.2). Then*

$$\langle H, c_{\Delta}(\kappa) \rangle = \sum \beta_i \kappa_i.$$

Proof. Assume that $F_i \sim F_j$. By definition, for all $\kappa \in \mathcal{C}_\Delta$ there exists an affine transformation $a_\kappa: \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ so that a_κ takes $\Delta(\kappa)$ to itself and takes F_i to F_j . Since $a_\kappa(F_i) = F_j$,

$$\langle \eta_i, x \rangle - \kappa_i = \langle \eta_j, a_\kappa(x) \rangle - \kappa_j \quad \forall x \in \mathfrak{t}^*;$$

see Remark 1.13. On the other hand, since $a_\kappa(\Delta(\kappa)) = \Delta(\kappa)$, a_κ fixes the center of mass $c_\Delta(\kappa)$. Hence,

$$\langle \eta_i - \eta_j, c_\Delta(\kappa) \rangle = \kappa_i - \kappa_j.$$

Finally, every inessential H is a linear combination of terms of the form $\eta_i - \eta_j$, where $F_i \sim F_j$. \square

Note that, in general, even if $H = \sum_i \beta_i \eta_i$ is mass linear the function $\langle H, c_\Delta(\kappa) \rangle$ need not equal $\sum \beta_i \kappa_i$; indeed, the coefficients of the η_i are not uniquely determined by H while the coefficients of κ_i are. In contrast, as we shall see in Lemma 2.6, if H is mass linear and $\langle H, c_\Delta(\kappa) \rangle = \sum \alpha_i \kappa_i$ then $H = \sum \alpha_i \eta_i$.

In §3, we analyze polytopes with inessential functions. In particular, we show that there are exactly two types of polytopes which admit inessential functions: bundles over the k -simplex (see Definition 3.10) and k -fold expansions (see Definition 3.13). We then show that it is possible to use Proposition 1.18 to reduce the number of asymmetric facets that we need to consider; we can subtract an inessential function H' from a mass linear function H to find a new mass linear function $\tilde{H} = H - H'$ with fewer asymmetric facets. For example, if H is a mass linear function on a simple polytope Δ then, after possibly subtracting an inessential function, we may assume that every asymmetric facet is pervasive (Proposition 3.25). This has the following consequence.

Proposition 1.19. *If Δ is a simple polytope that contains no pervasive facets then every mass linear function on Δ is inessential.*

In Example 1.16, we saw that every affine function on a product of simplices is mass linear. Our next theorem shows that these are the only polytopes with this property. The equivalence of conditions (iii) and (iv) below is reminiscent of Nill's Proposition 2.18 in [13]. However because Nill works only with rational polytopes the results are somewhat different.

Theorem 1.20. *Let Δ be a simple (or smooth) polytope. The following conditions are equivalent:*

- (i) *every $H \in \mathfrak{t}$ is mass linear on Δ ;*
- (ii) *every $H \in \mathfrak{t}$ is inessential on Δ ;*
- (iii) *Δ is a product of simplices;*
- (iv) *$\bigcap_{i \in I} F_i = \emptyset$ for every equivalence class of facets I ; in particular, $|I| > 1$.*

In §4, we analyze mass linear functions on low dimensional polytopes. In particular, we prove Theorems 1.3 and 1.4.

In part II, we will give a complete list 4-dimensional polytopes which admit essential mass linear functions. As described above, the main technique is to use “induction” by looking at lower dimensional symmetric faces. In the Appendix, in joint work with Timorin, we prove the key result that will enable us to begin this process.

Theorem 1.21. *Let $H \in \mathfrak{t}$ be a mass linear function on a smooth 4-dimensional polytope $\Delta \subset \mathfrak{t}^*$. Then there exists an inessential function $H' \in \mathfrak{t}$ so that the mass linear function $\tilde{H} = H - H'$ has the following property: at least one facet of Δ is \tilde{H} -symmetric.*

To prove this, we show that every 4-dimensional simple polytope which admits a mass linear function with no symmetric facets is combinatorially equivalent to the product of simplices.

1.2. Geometric motivation. Our motivation for studying mass linear functions on polytopes is geometric; we wish to understand the fundamental group of the group of symplectomorphisms of a symplectic toric manifold. For other approaches to this question see Januszkiewicz–Kędra [10], Kędra–McDuff [11], McDuff–Tolman [12], and Viña [17]. Additionally, Entov–Polterovich [8] give some interesting applications of compressible subtori of the group of symplectomorphisms; see Definition 1.23.

Let (M, ω, T, Φ) be a symplectic toric manifold, where M is a compact connected manifold of dimension $2n$, ω is a symplectic form on M , $T = \mathfrak{t}/\ell$ is a compact n -dimensional torus, and $\Phi: M \rightarrow \mathfrak{t}^*$ is a moment map for an effective T action on (M, ω) . Then the **moment polytope** $\Delta = \Phi(M) \subset \mathfrak{t}^*$ is a smooth polytope. Conversely, every smooth polytope $\Delta \subset \mathfrak{t}^*$ determines a canonical symplectic toric manifold $(M_\Delta, \omega_\Delta, \Phi_\Delta)$ with moment polytope $\Delta = \Phi_\Delta(M_\Delta)$. (See §5 for more details.) Finally, any two symplectic toric manifolds with the same moment polytope are equivariantly symplectomorphic [6].

For example, the symplectic manifold associated to the standard n -simplex Δ_n is complex projective space $\mathbb{C}P^n$ with the standard (i.e. diagonal) $(S^1)^n$ action and suitably normalized Fubini-Study form. More generally, a product of simplices corresponds to a product of projective spaces. Similarly, the polytopes constructed in Example 1.1 correspond to $\mathbb{C}P^2$ bundles over $\mathbb{C}P^1$; cf. Example 5.3.

Let $\text{Symp}_0(M, \omega)$ denote the identity component of the group of symplectomorphisms of (M, ω) . Since the torus T is a subgroup of $\text{Symp}_0(M, \omega)$, each $H \in \ell$ induces a circle $\Lambda_H \subset T \subset \text{Symp}_0(M, \omega)$. More generally, each $H \in \mathfrak{t}$ generates a Hamiltonian \mathbb{R} -action on M whose closure is a subtorus $T_H \subset T \subset \text{Symp}_0(M, \omega)$. The next proposition is proved in §5 by interpreting the quantity $\langle H, c_\Delta(\kappa) \rangle$ as the value of Weinstein’s action homomorphism $\mathcal{A}_H: \pi_1(\text{Symp}_0(M_\Delta, \omega_\Delta)) \rightarrow \mathbb{R}/\mathcal{P}_\omega$.

Proposition 1.22. *Let (M, ω, T, Φ) be a symplectic toric manifold with moment polytope $\Delta \subset \mathfrak{t}^*$; fix $H \in \ell$. If the image of Λ_H in $\pi_1(\text{Symp}_0(M, \omega))$ is trivial, then H is a mass linear function with integer coefficients, that is, there exist integers $\alpha_1, \dots, \alpha_N$ so that*

$$\langle H, c_\Delta(\kappa) \rangle = \sum \alpha_i \kappa_i$$

for all κ in the chamber \mathcal{C}_Δ .

Definition 1.23. *Given a topological group G , a subtorus $K \subset G$ is **compressible** in G if the natural map $\pi_1(K) \rightarrow \pi_1(G)$ has finite image.*

Corollary 1.24. *Let (M, ω, T, Φ) be a symplectic toric manifold with moment polytope $\Delta \subset \mathfrak{t}^*$. If $K \subset T$ is compressible in $\text{Symp}_0(M, \omega)$, then every $H \in \mathfrak{k} \subset \mathfrak{t}$ is mass linear.*

Theorem 1.25. *Let (M, ω, T, Φ) be a symplectic toric manifold with moment polytope Δ .*

- (i) *The map $\pi_1(T) \rightarrow \pi_1(\text{Symp}_0(M, \omega))$ is an injection if there are no mass linear functions $H \in \ell$ on Δ .*
- (ii) *The torus T is compressible in $\text{Symp}_0(M, \omega)$ exactly if (M, ω) is a product of projective spaces with a product symplectic form. In particular, the image of $\pi_1(T)$ in $\pi_1(\text{Symp}_0(M, \omega))$ is never zero.*

Proof. Part (i) is an immediate consequence of Proposition 1.22. The first claim of part (ii) follows by combining Theorem 1.20 with Corollary 1.24. The second claim follows from Proposition 1.22 and Example 1.16. \square

Remark 1.26. More generally, since “most” polytopes have no nonconstant mass linear functions, the above theorem implies that in most cases the map $\pi_1(T) \rightarrow \pi_1(\text{Symp}_0(M, \omega))$ is injective. For example, in the 4-dimensional case Theorem 1.3 implies that it is injective if Δ has five or more facets, while in the 6-dimensional case Theorem 4.13 implies that it is injective unless M is a bundle over a projective space or is a smooth pencil of 4-dimensional toric manifolds that intersect along a $\mathbb{C}P^1$. (Here, we are using Remark 5.2 and Remark 5.4.) Further, for each facet F_i , let $x_i \in H^2(M; \mathbb{Z})$ denote the Poincaré dual to the divisor $\Phi^{-1}(F_i)$. Then Theorem 1.11 implies the following: the map is injective if, for each i , $x_i^2 \neq 0$ but there is $j \neq i$ such that $x_i \cdot x_j = 0$. Thus we can always make the map injective by equivariantly blowing M up at a suitable number of fixed points.

Next, we claim that, given any smooth polytope $\Delta \subset \mathfrak{t}^*$, there is also a canonical complex structure J_Δ on the associated symplectic toric manifold M_Δ ; this complex structure is T -equivariant and is compatible with ω_Δ . By the discussion above, this implies that there is a compatible complex structure J on every symplectic toric manifold (M, ω, T, Φ) ; moreover, up to T -equivariant symplectomorphism, this complex structure is canonical. For example, the canonical complex structure on $\mathbb{C}P^n$ is the standard one.

If there are robust symmetries of the moment polytope $\Delta = \Phi(M)$, then there are extra symmetries of (M, ω) which preserve the canonical Kähler structure (ω, J, g_J) , where g_J denotes the associated metric $\omega(\cdot, J\cdot)$. We shall see in Proposition 5.5 that these robust symmetries in some sense generate $\text{Isom}_0(M)$, the identity component of the group of isometries of the Riemannian manifold (M, g_J) . Moreover, $\text{Isom}_0(M)$ is a maximal connected compact Lie subgroup of $\text{Symp}_0(M, \omega)$. For example, if $M = \mathbb{C}P^k$, $\text{Isom}_0(M)$ is the projective unitary group $PU(k+1)$. Finally, note that T is also a subgroup of $\text{Isom}_0(M)$.

Lemma 1.27. *Let (M, ω, T, Φ) be a symplectic toric manifold, and let $\text{Isom}_0(M)$ be the identity component of the associated Kähler isometry group. The map $\pi_1(T) \rightarrow \pi_1(\text{Isom}_0(M))$ is surjective.*

Moreover, let $\Delta = \bigcap_{i=1}^N \{x \in \mathfrak{t}^* \mid \langle \eta_i, x \rangle \leq \kappa_i\}$ be the moment polytope. Let \mathcal{I} denote the set of equivalence classes of facets of Δ , and fix $H \in \ell$. The image of Λ_H in $\pi_1(\text{Isom}_0(M))$ is trivial exactly if H can be written

$$H = \sum \beta_i \eta_i, \quad \text{where } \beta_i \in \mathbb{Z} \ \forall i \text{ and } \sum_{i \in I} \beta_i = 0 \ \forall I \in \mathcal{I}.$$

The above lemma motivated the definition of inessential function. Its proof, given in §5, is independent of the intervening sections, except for the discussion of the equivalence relation in §3.1.

Corollary 1.28. *Let (M, ω, T, Φ) be a symplectic toric manifold, and let $\text{Isom}_0(M)$ be the identity component of the associated Kähler isometry group. Given $H \in \mathfrak{t}$, the torus T_H is compressible in $\text{Isom}_0(M)$ exactly if H is inessential on Δ .*

Proof. The Lie algebra \mathfrak{t}_H of T_H is the smallest rational subspace of \mathfrak{t} containing H . Since the set of inessential $H' \in \mathfrak{t}$ is clearly a rational subspace of \mathfrak{t} , this implies that H is mass linear exactly if every $H' \in \ell \cap \mathfrak{t}_H$ is mass linear. Therefore it suffices to prove the result for $H \in \ell$. But by Lemma 3.9, $H \in \ell$ is inessential exactly if H can be written $H = \sum \beta_i \eta_i$, where $\beta_i \in \mathbb{Q}$ for all i and $\sum_{i \in I} \beta_i = 0$ for all $I \in \mathcal{I}$. By Lemma 1.27, this implies that H is inessential exactly if there exists a natural number m such that Λ_{mH} contracts in $\text{Isom}_0(M)$, that is, exactly if Λ_H has finite order in $\text{Isom}_0(M)$. \square

Our final result concerns the map

$$\rho : \pi_1(\text{Isom}_0(M)) \rightarrow \pi_1(\text{Symp}_0(M, \omega))$$

induced by inclusion.

Proposition 1.29. *Let (M, ω, T, Φ) be a symplectic toric manifold, and let $\text{Isom}_0(M)$ be the identity component of the associated Kähler isometry group. If there are no essential mass linear functions $H \in \ell$ on the moment polytope $\Delta \subset \mathfrak{t}^*$, then the map $\rho : \pi_1(\text{Isom}_0(M)) \rightarrow \pi_1(\text{Symp}_0(M, \omega))$ is an injection.*

Proof. Assume that ρ is not injective. By Lemma 1.27, the map $\pi_1(T) \rightarrow \pi_1(\text{Isom}_0(M))$ is surjective. Therefore, there exists $H \in \ell$ such that the image of Λ_H in $\pi_1(\text{Symp}_0(M, \omega))$ vanishes but the image of Λ_H in $\pi_1(\text{Isom}_0(M))$ does not. By Proposition 1.22, H is mass linear; moreover, $\langle H, c_\Delta \rangle = \sum \alpha_i \kappa_i$, where $\alpha_i \in \mathbb{Z}$ for all i .

Since we have assumed that Δ has no essential mass linear functions, H must be inessential. Hence, we may write $H = \sum \beta_i \eta_i$ where $\sum_{i \in I} \beta_i = 0$ for each equivalence class of facets I . By Proposition 1.18 we must have $\langle H, c_\Delta(\kappa) \rangle = \sum \beta_i \kappa_i$. But then $\beta_i = \alpha_i$ because the variables κ_i are independent. In particular $\beta_i \in \mathbb{Z}$. Lemma 1.27 then implies that the image of Λ_H in $\pi_1(\text{Isom}_0(M))$ vanishes. This gives a contradiction. \square

Example 1.30. If $M = \mathbb{C}P^k$, then $H = e_1$ generates a circle Λ_H with order $k + 1$ in $\pi_1(\text{Isom}_0(\mathbb{C}P^k)) = \pi_1(PU(k + 1)) = \mathbb{Z}/(k + 1)$. It follows from Proposition 1.22 (and Example 1.15) that Λ_H also has order $k + 1$ in $\pi_1(\text{Symp}_0(\mathbb{C}P^k))$, a result first proved in Seidel [15]; see also [12].

Remark 1.31. The converse of Proposition 1.22 is open, that is, we do not know if the following statement holds:

If $H \in \ell$ is a mass linear function with integer coefficients, then the image of Λ_H in $\pi_1(\text{Symp}_0(M_\Delta, \omega_\Delta))$ is trivial.

This seems very likely to be the case, at least in dimension 3, and is the subject of ongoing research.

If the statement above did hold, it would imply that the converse to part (i) of Theorem 1.25 is also true, and hence that

The map $\rho : \pi_1(\text{Isom}_0(M)) \rightarrow \pi_1(\text{Symp}_0(M, \omega))$ is an injection exactly if there are no essential mass linear functions $H \in \ell$ on Δ .

To see this, note that if there were an essential mass linear $H \in \ell$ then by Remark 2.4 it would have rational coefficients. Hence some multiple would have integer coefficients. By our assumption and by Lemma 1.27, this would give an element of $\pi_1(T)$ which lies in the kernel of the homomorphism $\pi_1(T) \rightarrow \pi_1(\text{Symp}_0(M, \omega))$ but does not lie in the kernel of the homomorphism $\pi_1(T) \rightarrow \pi_1(\text{Isom}_0(M))$.

Example 1.32. Let M be the one point blow up of \mathbb{CP}^3 . Then $\text{Isom}_0(M) = U(3)$, and Theorem 1.5 implies that $\pi_1(U(3)) = \mathbb{Z}$ injects into $\pi_1(\text{Symp}_0(M, \omega))$, a result already noted by Viña [17]. In this case, we also understand the behavior of the higher homotopy groups, at least rationally; $\pi_3(U(3)) = \mathbb{Z}$ injects into $\pi_3(\text{Symp}_0(M, \omega))$ by Reznikov [14] (cf. also [11]), while $\pi_5(U(3)) \otimes \mathbb{Q}$ injects into $\pi_5(\text{Symp}_0(M, \omega)) \otimes \mathbb{Q}$ because it has nontrivial image in $\pi_5(\mathbb{CP}^3) \otimes \mathbb{Q}$ under the composite of the point evaluation map $\text{Symp}_0(M, \omega) \rightarrow M$ with the blow down $M \rightarrow \mathbb{CP}^3$.

Remark 1.33. If M is a \mathbb{CP}^2 bundle over \mathbb{CP}^1 , the map $\pi_1(T) \rightarrow \pi_1(\text{Isom}_0(M))$ is not injective: the action of $SU(2)$ on the base \mathbb{CP}^1 lifts to an action of $SU(2)$ on M that contains one of the circle factors in T . On the other hand, for generic toric manifolds $\text{Isom}_0(M) = T$. One might wonder whether the condition $\text{Isom}_0(M) = T$ is enough to imply that $\pi_1(T)$ injects into $\pi_1(\text{Symp}_0(M, \omega))$. However, this seems unlikely to be true since, as we shall see in Part II, there are mass linear pairs (H, Δ) where H is not constant and Δ has no robust symmetries, i.e. is such that $\text{Isom}_0(M) = T$. In this case, although Λ_H may represent a nonzero element in $\pi_1(\text{Symp}_0(M, \omega))$, our methods are not sufficiently powerful to detect this.

Remark 1.34. There seems to be very little work on simple polytopes that mentions the center of mass. A notable exception is Nill's paper [13] on complete toric varieties. Interestingly enough, this paper is also concerned with questions about the symmetries of M , but considers the full group of biholomorphisms $\text{Aut}(M)$ rather than $\text{Isom}_0(M)$, which is one of its maximal connected compact subgroups. His work applies in the special case of smooth Fano varieties. The moment polytope Δ of such a variety contains a special point p_Δ . Geometers usually normalize the polytope so that p_Δ lies at the origin but this point can be described purely in terms of the moment polytope; see Entov–Polterovich [8, Prop. 1.6]. Nill gives a direct combinatorial proof of the fact that if a Fano moment polytope Δ has the property that p_Δ is equal to the center of mass c_Δ of Δ , then $\text{Aut}(M)$ is reductive. (For terminology, see Remark 5.8 below.) In fact, the difference between p_Δ and c_Δ is precisely the Futaki invariant, which is now known to vanish if and only if (M, J) supports a Kähler–Einstein metric; cf. Wang–Zhu [18]. Moreover the automorphism group of any Kähler–Einstein manifold is reductive; this gives an alternate proof of Nill's result.

However, the *existence* of mass linear functions, which is the question that concerns us here, is not related in any simple way to the vanishing of the Futaki invariant. For example, the Futaki invariant of the blow up of $\mathbb{C}P^2$ at three points vanishes. But the corresponding polytope supports no nontrivial mass linear functions by Theorem 1.3 (or by Theorem 1.11). Thus $\pi_1(T^2)$ injects into $\pi_1(\text{Symp}_0(M, \omega))$ in this case.

Organization of the paper. The main results in this paper are Theorems 1.3 and 1.4 that classify the low dimensional cases, Theorem 1.20 concerning the structure of polytopes for which all H are mass linear, and Theorems 1.5 and 1.25 which give geometric interpretations of these results. A more elaborate version of Theorem 1.3 is stated and proved as Proposition 4.2. Theorem 1.4 is proved in §4.4, and leads into Proposition 4.14 which classifies all 3-dimensional mass linear pairs. Theorem 1.20 is proved in §3.5, while all the geometric results are proved in §5.

Acknowledgments. We warmly thank Vladen Timorin for his important contribution to the Appendix. His ideas also helped us clarify some of the proofs in §2. We are also very grateful to the referee for a very careful reading and for the many suggestions that helped us improve the exposition.

2. BASIC IDEAS

In this section, we introduce the basic ideas which underlie the results in this paper and its sequel. More specifically, we first introduce the volume and moment functions and then use them to analyze the key features of symmetric faces and asymmetric facets. Additionally, we prove Theorem 1.11 in §2.4.

2.1. Notational conventions. Unless there is explicit mention to the contrary, we fix an identification of the affine space A with \mathfrak{t}^* and hence assume throughout that

$$(2.1) \quad \Delta = \bigcap_{i=1}^N \{x \in \mathfrak{t}^* \mid \langle \eta_i, x \rangle \leq \kappa_i\}$$

is a simple polytope with facets F_1, \dots, F_N and outward conormals η_1, \dots, η_N . Further, for each $I \subset \{1, \dots, N\}$, we denote the intersection of the corresponding facets by

$$(2.2) \quad F_I := \bigcap_{i \in I} F_i.$$

We will need to understand the faces of a polytope as polytopes in their own right. To this end, fix a simple polytope Δ . Given a (nonempty) face $f = F_I$, let $P(f) \subset \mathfrak{t}^*$ denote the smallest affine subspace which contains f . Since Δ is simple, $P(f)$ has codimension $|I|$; indeed,

$$(2.3) \quad P(f) = \bigcap_{i \in I} \{x \in \mathfrak{t}^* \mid \langle \eta_i, x \rangle = \kappa_i\}.$$

Additionally, if F is a facet of Δ which does not contain f , then $F \cap f$ is either empty or is a facet of f ; conversely, every facet of f has this form. Therefore,

$$f = \bigcap_{j \in I_f} \{x \in P(f) \mid \langle \eta_j, x \rangle \leq \kappa_j\},$$

where $j \in \{1, \dots, N\}$ lies in I_f exactly if $j \notin I$ and $F_j \cap f \neq \emptyset$.

Let $\widehat{\mathfrak{t}}$ denote the quotient of \mathfrak{t} by the span of the $\{\eta_i\}_{i \in I}$, and let $\pi: \mathfrak{t} \rightarrow \widehat{\mathfrak{t}}$ be the natural projection. Fix $x \in P(f)$ and define an isomorphism $j_x: \widehat{\mathfrak{t}}^* \rightarrow P(f)$ by $j_x(y) = \pi^*(y) + x$. Then

$$(2.4) \quad f_x := j_x^{-1}(f) = \bigcap_{j \in I_f} \{y \in \widehat{\mathfrak{t}}^* \mid \langle \pi(\eta_j), y \rangle \leq \kappa_j - \langle \eta_j, x \rangle\};$$

in particular, $f_x \subset \widehat{\mathfrak{t}}^*$ is a convex polytope with outward conormals $\{\pi(\eta_j)\}_{j \in I_f}$. With this explicit description of the faces of Δ , it is then easy to establish the following important facts:

- Every face of a simple polytope is itself a simple polytope.
- Every face of a smooth polytope is itself a smooth polytope.

2.2. The volume and moment of Δ . The proofs in this section were greatly influenced by the point of view in Timorin [16].

Fix $H \in \mathfrak{t}$ and a simple polytope $\Delta \subset \mathfrak{t}^*$. Fix an identification of \mathfrak{t}^* with Euclidean space, and consider the following polynomials of the support numbers:

$$V = \int_{\Delta} dx \quad \text{and} \quad \mu = \int_{\Delta} H(x) dx.$$

That is, V is the volume of Δ and μ is the first moment of Δ in the direction given by H ; on \mathcal{C}_{Δ} they are polynomials of degree at most $\dim \Delta$ and $\dim \Delta + 1$ respectively.

More generally, for any face f of Δ , fix an identification of $P(f)$ with Euclidean space. Let V_f denote the volume of $f \subset P(f)$; analogously, let μ_f denote the integral of the function H over the face f . On \mathcal{C}_{Δ} , V_f and μ_f are polynomials of degree at most $\dim f$ and $\dim(f) + 1$ respectively. Let ∂_i denote the operator of differentiation with respect to κ_i .

Remark 2.1. If Δ is a smooth polytope, then it is natural to choose identifications of \mathfrak{t}^* and $P(f)$ which respect the lattice. If we make this choice, the constant K_f in the proposition below is 1 for every non-empty face.

However, if Δ is not rational then $P(f)$ has no lattice and there is no preferred identification with Euclidean space. While the center of mass of f does not depend on this choice, the constant K_f in the proposition below does.

Proposition 2.2. *Let f be the intersection of the facets F_{i_1}, \dots, F_{i_k} . Then*

$$\partial_{i_1} \cdots \partial_{i_k} V = K_f V_f \quad \text{and} \quad \partial_{i_1} \cdots \partial_{i_k} \mu = K_f \mu_f,$$

where K_f is a positive real number that depends only on f . In particular, if the intersection of facets F_{i_1}, \dots, F_{i_k} is empty, then

$$\partial_{i_1} \cdots \partial_{i_k} V = \partial_{i_1} \cdots \partial_{i_k} \mu = 0.$$

Proof. Let $f = F_1 \cap \cdots \cap F_k$. Let e_1, \dots, e_n denote the standard basis of \mathbb{R}^n , where $n = \dim \mathfrak{t}$.

If f is not empty, then η_1, \dots, η_k are linearly independent. Moreover, changing the identifications of \mathfrak{t}^* and $P(f)$ with Euclidean space simply alters the measure by a positive constant; this only affects the value of K_f . Hence, we may identify \mathfrak{t}^* with \mathbb{R}^n so that η_1, \dots, η_k are identified with e_1, \dots, e_k , and thus identify $P(f)$ with $\{0\} \times \mathbb{R}^{n-k}$. We can write V as the integral

$$V = \int_{-\infty}^{\kappa_k} \cdots \int_{-\infty}^{\kappa_1} V(x_1, \dots, x_k) dx_1 \cdots dx_k,$$

where $V(c_1, \dots, c_k)$ is the volume of $\Delta \cap (\{c\} \times \mathbb{R}^{n-k})$ where $c = (c_1, \dots, c_k)$. Therefore, $\partial_1 \cdots \partial_k V = V(\kappa_1, \dots, \kappa_k) = V_f$ by the fundamental theorem of calculus. Similarly, we can write μ as the integral

$$\mu = \int_{-\infty}^{\kappa_k} \cdots \int_{-\infty}^{\kappa_1} \mu(x_1, \dots, x_k) dx_1 \cdots dx_k,$$

where $\mu(c_1, \dots, c_k)$ is the integral of H over the polytope $\Delta \cap (\{c\} \times \mathbb{R}^{n-k})$. Hence, $\partial_1 \cdots \partial_k \mu = \mu(\kappa_1, \dots, \kappa_k) = \mu_f$.

So suppose instead that f is empty. Let $g = F_2 \cap \cdots \cap F_k$. We may assume by induction that $\partial_2 \cdots \partial_k V = K_g V_g$ for some positive constant K_g , and so $\partial_1 \cdots \partial_k V = K_g \partial_1 V_g$. However, since F_1 and g do not intersect, g does not depend on κ_1 . Therefore, $\partial_1 V_g = 0$. A similar argument shows that $\partial_1 \cdots \partial_k \mu = 0$. \square

Given $H \in \mathfrak{t}$ and a simple polytope $\Delta \subset \mathfrak{t}^*$, define a function $\widehat{H} : \mathcal{C}_\Delta \rightarrow \mathbb{R}$ by

$$(2.5) \quad \widehat{H}(\kappa_1, \dots, \kappa_N) = \langle H, c_\Delta(\kappa_1, \dots, \kappa_N) \rangle,$$

where c_Δ denotes the center of mass of Δ . We will repeatedly use the fact that

$$\mu = \widehat{H}V.$$

The next lemma is a simple consequence of the fact that, since V and μ are both polynomials, the function $\widehat{H} = \frac{\mu}{V}$ is a rational function.

Lemma 2.3. *Fix $H \in \mathfrak{t}$ and a simple polytope $\Delta \subset \mathfrak{t}^*$. If the function \widehat{H} defined above is linear on some open subset of \mathcal{C}_Δ , it is linear on all of \mathcal{C}_Δ .*

Remark 2.4. If $\Delta \subset \mathfrak{t}^*$ is a smooth polytope, then every mass linear function $H \in \ell$ has rational coefficients. (Recall that ℓ is the integer lattice in \mathfrak{t} .) In other words, if $\langle H, c_\Delta(\kappa) \rangle = \sum \gamma_i \kappa_i$ for all $\kappa \in \mathcal{C}_\Delta$ then $\gamma_i \in \mathbb{Q}$.

To see this, note that by Proposition 2.2 and Remark 2.1, the volume $V : \mathcal{C}_\Delta \rightarrow \mathbb{R}$ is a polynomial with rational coefficients. Similarly, since $\langle H, v \rangle : \mathcal{C}_\Delta \rightarrow \mathbb{R}$ is a linear function with integer coefficients for every vertex $v \in \Delta$, the moment μ is also a polynomial with rational coefficients. So $\frac{\mu}{V}$ is a rational function with rational coefficients.

We include the following lemma for completeness, but make no use of it in this paper.

Lemma 2.5. *Let $\Delta \subset \mathfrak{t}^*$ be a smooth polytope. Then the set of mass linear functions on Δ is a rational subspace of \mathfrak{t} .*

To prove it, we will need the following useful lemma, which shows that the coefficients of a mass linear function $H \in \mathfrak{t}$ determine the function H itself; cf. Proposition 1.18.

Lemma 2.6. *Fix $H \in \mathfrak{t}$ and a simple polytope $\Delta \subset \mathfrak{t}^*$. If $\langle H, c_\Delta(\kappa) \rangle = \sum \beta_i \kappa_i$, then*

$$H = \sum \beta_i \eta_i.$$

Proof. Given any $\xi \in \mathfrak{t}^*$, let $\Delta' = \Delta + \xi$, the translate of Δ by ξ . Then

$$(2.6) \quad \Delta' = \Delta(\kappa') := \bigcap_{i=1}^N \{x \in \mathfrak{t}^* \mid \langle \eta_i, x \rangle \leq \kappa'_i\}, \quad \text{where} \quad \kappa'_i = \kappa_i + \langle \eta_i, \xi \rangle \quad \forall i.$$

Hence, by assumption

$$\begin{aligned} \langle H, c_\Delta(\kappa') \rangle &= \sum \beta_i \kappa'_i \\ &= \sum \beta_i \kappa_i + \sum \beta_i \langle \eta_i, \xi \rangle \\ &= \langle H, c_\Delta(\kappa) \rangle + \sum \langle \beta_i \eta_i, \xi \rangle. \end{aligned}$$

Since clearly $\langle H, c_\Delta(\kappa') \rangle = \langle H, c_\Delta(\kappa) \rangle + \langle H, \xi \rangle$, this implies that

$$\langle H, \xi \rangle = \sum \langle \beta_i \eta_i, \xi \rangle \quad \forall \xi \in \mathfrak{t}^*,$$

as required. \square

Proof of Lemma 2.5 For all $i \in \{1, \dots, N\}$, let μ_i denote the moment in the η_i direction. By the previous remark, the polynomial $\mu_i - V\kappa_i$ is a polynomial with rational coefficients for all $i \in \{1, \dots, N\}$. Therefore, the set

$$X = \left\{ \beta \in \mathbb{R}^N \mid \sum_{i=1}^N \beta_i \mu_i = \sum_{i=1}^N \beta_i V \kappa_i \right\}$$

is a rational subspace of \mathbb{R}^N . Hence, to prove the result, it is enough to show that $H \in \mathfrak{t}$ is mass linear exactly if $H = \sum \beta_i \eta_i$, where $\beta \in X$. To check this, note that $\mu_i/V = \langle \eta_i, c_\Delta(\kappa) \rangle$. Hence

$$\left\langle \sum \beta_i \eta_i, c_\Delta(\kappa) \right\rangle = \sum \beta_i \frac{\mu_i}{V}.$$

Therefore, if $\beta \in X$ then $\langle \sum \beta_i \eta_i, c_\Delta(\kappa) \rangle = \sum \beta_i \kappa_i$. Conversely, consider $H \in \mathfrak{t}$ so that $\langle H, c_\Delta(\kappa) \rangle = \sum \beta_i \kappa_i$. By Lemma 2.6, this implies that $H = \sum \beta_i \eta_i$. But then

$$\sum \langle \beta_i \eta_i, c_\Delta(\kappa) \rangle = \sum \beta_i \kappa_i,$$

which implies that $\beta \in X$. \square

2.3. Symmetric faces. We are now ready to consider symmetric faces; see Definition 1.9. In particular, we will analyze the restriction of a mass linear function to a symmetric face.

Lemma 2.7. *Fix $H \in \mathfrak{t}$ and let f be a symmetric face of a simple polytope $\Delta \subset \mathfrak{t}^*$. Then*

$$\langle H, c_f(\kappa) \rangle = \langle H, c_\Delta(\kappa) \rangle \quad \forall \kappa \in \mathcal{C}_\Delta,$$

where $c_f: \mathcal{C}_\Delta \rightarrow \mathfrak{t}^*$ denotes the center of mass of f .

Proof. Let $f = F_J$ be a symmetric face; number the facets so that $J = \{1, \dots, j\}$. Since F_i is symmetric for each $1 \leq i \leq j$, $\partial_i \hat{H} = 0$. Hence, by Proposition 2.2, if we apply the differential operator $\partial_1 \cdots \partial_j$ to the formula $\mu = \hat{H}V$ we obtain

$$K_f \mu_f = K_f \hat{H} V_f.$$

Since $K_f \neq 0$ and $\mu_f = \langle H, c_f \rangle V_f$, this proves that $\langle H, c_\Delta \rangle = \langle H, c_f \rangle$. \square

Lemma 2.8. *Fix $H \in \mathfrak{t}$ and let $\Delta \subset \mathfrak{t}^*$ be a simple polytope. If f is a symmetric face and F an asymmetric facet, then $f \cap F \neq \emptyset$; indeed, $F \cap f$ is a facet of f .*

Proof. Suppose on the contrary that $F \cap f = \emptyset$. Since $f(\kappa) \subset \Delta(\kappa)$ does not depend on support number of F ; neither does $\langle H, c_f(\kappa) \rangle: \mathcal{C}_\Delta \rightarrow \mathbb{R}$. By the lemma above, this implies that $\langle H, c_\Delta(\kappa) \rangle: \mathcal{C}_\Delta \rightarrow \mathbb{R}$ does not depend on support number of F , that is, F is symmetric. This is a contradiction. Finally, we note that by Definition 1.9, $f \not\subset F$. Hence $F \cap f$ must be a facet of f . \square

Proposition 2.9. *Let $H \in \mathfrak{t}$ be a mass linear function on a simple polytope $\Delta \subset \mathfrak{t}^*$. Let f be a symmetric face of Δ . Then the restriction of H to f is mass linear and the map $F \mapsto F \cap f$ induces a one-to-one correspondence between the asymmetric facets of Δ and the asymmetric facets of f . Moreover, the coefficient of the support number of F in $\langle H, c_\Delta \rangle$ is the coefficient of the support number of $f \cap F$ in $\langle H, c_f \rangle$.*

Proof. Renumber the facets so that $F_j \cap f$ is a facet of f exactly if $j \in \{1, \dots, M\}$. In the statement of this proposition, we consider f as a polytope in its own right, that is, $f = \bigcap_{j=1}^M \{x \in P(f) \mid \langle \eta_j, x \rangle \leq \kappa_j\}$, where $P(f)$ denotes the affine span of f as in Equation (2.3). Hence, we consider the center of mass of f as a function on $\mathcal{C}_f \subset \mathbb{R}^M$. In contrast, in Lemma 2.7, we consider f as a face of the polytope Δ and consider c_f as a function on $\mathcal{C}_\Delta \subset \mathbb{R}^N$. However, if F is an asymmetric facet of Δ then by Lemma 2.8 $F \cap f$ is a facet of f . Thus the facets F_i for $i > M$ are symmetric, so that $\langle H, c_\Delta \rangle: \mathcal{C}_\Delta \rightarrow \mathfrak{t}^*$ does not depend on the support number κ_i . Therefore, Lemma 2.7 implies that

$$(2.7) \quad \langle H, c_f(\kappa_1, \dots, \kappa_M) \rangle = \langle H, c_\Delta(\kappa_1, \dots, \kappa_N) \rangle \quad \forall \kappa \in \mathcal{C}_\Delta.$$

Next observe that the set of M -tuples $(\kappa_1, \dots, \kappa_M)$ that extend to N -tuples $(\kappa_1, \dots, \kappa_N) \in \mathcal{C}_\Delta$ forms an open subset U of \mathcal{C}_f that may not be the whole of \mathcal{C}_f . Nevertheless, by Lemma 2.3 the fact that the function $\langle H, c_f \rangle$ is linear on U implies that it is linear everywhere in \mathcal{C}_f . Moreover its coefficients equal those of $\langle H, c_\Delta \rangle$. Since a facet is asymmetric if and only if the corresponding coefficient is nonzero, the result is now immediate. \square

Remark 2.10. The following notion is sometimes useful. Fix $H \in \mathfrak{t}$ and a simple polytope $\Delta \subset \mathfrak{t}^*$. We say that a face f of Δ is **centered** if

$$\langle H, c_f(\kappa) \rangle = \langle H, c_\Delta(\kappa) \rangle \quad \forall \kappa \in \mathcal{C}_\Delta.$$

Lemma 2.7 shows that symmetric faces are centered. Conversely, if F is a centered facet, then by definition F is also symmetric. In contrast, the converse does not hold for faces of higher codimension. For example, let $\Delta_2 = \{x \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0, \text{ and } x_1 + x_2 \leq 1\}$ denote the standard 2-simplex, and let $H(x) = x_1 - x_2$. Then the vertex $(0, 0)$ is centered, but it is not symmetric because the facets containing it are not symmetric. The proof of Lemma 2.8 can be adapted to show that every centered face must intersect every asymmetric facet.

2.4. Asymmetric facets. We now consider asymmetric facets. The main goal of this subsection is to prove Theorem 1.11 and the proposition below; see Definition 1.10.

Proposition 2.11. *Let $H \in \mathfrak{t}$ be a mass linear function on a simple polytope $\Delta \subset \mathfrak{t}^*$. Then every asymmetric facet is pervasive or flat (or both).*

We will need the following lemmas.

Lemma 2.12. *Let $\Delta \subset \mathfrak{t}^*$ be a simple polytope. The facet F_i is flat if and only if the volume V is a linear function of κ_i .*

Proof. If F_i is flat, a straightforward computation shows that the volume is a linear function of κ_i . So assume that F_i is not flat. At each vertex v of F_i there is a unique edge e_v of Δ that does not lie in F_i . Since F_i is not flat, these edges cannot all be parallel. In particular, there is an edge e joining two vertices v and v' so that e_v is not parallel to $e_{v'}$. Write $e = F_J \cap F_i$ for some $J = \{j_1, \dots, j_{n-2}\} \subset \{1, \dots, N\}$. Since the edges e_v and $e_{v'}$ of the polygon F_J are not parallel, V_{F_J} is not a linear function of κ_1 ; it has degree 2. But by Proposition 2.2, $\partial_{j_1} \cdots \partial_{j_{n-2}} V = K_{F_J} V_{F_J}$. Hence, V is not a linear function of κ_1 . \square

Lemma 2.13. *Fix a nonzero $H \in \mathfrak{t}$ and a simple polytope $\Delta \subset \mathfrak{t}^*$. Then Δ has at least two asymmetric facets.*

Proof. Let F_i be any facet. Since Δ has a nonempty interior, c_Δ does not lie on F_i . Therefore, by translating and then rescaling, we can find a dilation of \mathfrak{t}^* that fixes the plane $P(F_i)$ but moves c_Δ to an arbitrary nearby point. Hence $\langle H, c_\Delta \rangle$ depends on at least one κ_j for $j \neq i$. \square

Proof of Proposition 2.11. Let F_1 be an asymmetric facet which is not flat and let F_2 be a facet which does not meet F_1 . Since H is mass linear, the function \hat{H} of Equation (2.5) is linear, and so its second derivatives all vanish. Hence, since $F_1 \cap F_2 = \emptyset$, by Proposition 2.2 if we apply the differential operator $\partial_1 \partial_2$ to the formula $\mu = \hat{H}V$ we obtain

$$0 = (\partial_1 \hat{H})V_{F_2} + (\partial_2 \hat{H})V_{F_1}.$$

Applying the operator ∂_1 again to the equation above, we obtain

$$(\partial_2 \hat{H}) \partial_1 V_{F_1} = 0.$$

On the one hand, Lemma 2.8 implies that F_2 is asymmetric. Thus \widehat{H} depends linearly on the support number of F_2 , so that $\partial_2 \widehat{H} \neq 0$. On the other hand, since F_1 is not flat Lemma 2.12 implies that $\partial_1 V_{F_1} \neq 0$; this gives a contradiction. \square

Proof of Theorem 1.11. Suppose that the simple polytope $\Delta \subset \mathfrak{t}^*$ contains no pervasive facets and no flat facets. It suffices to show that every homogeneous function $H \in \mathfrak{t}$ that is mass linear on Δ must vanish. But if such an H is nonzero, then Δ contains an asymmetric facet F by Lemma 2.13. Moreover, F must be pervasive or flat by Proposition 2.11, which is impossible. \square

3. POLYTOPES WITH INESSENTIAL FUNCTIONS

The main results of this section are Proposition 3.17, which characterizes polytopes with nonconstant inessential functions, and Theorem 1.20, which characterizes polytopes with the property that all linear functions are mass linear.

3.1. Symmetries and equivalent facets. In this subsection, we prove several useful criteria for determining if facets are equivalent. To begin, we justify our normalization conventions by proving a few easy facts about symmetries: see Remark 1.13. Because this is not yet accomplished, we will **not** assume that our conormals are normalized in Lemma 3.1, though we do make this assumption in subsequent arguments.

Lemma 3.1. *Let $a: \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ be a symmetry of a simple polytope $\Delta \subset \mathfrak{t}$. Let $\widehat{a}: \mathfrak{t} \rightarrow \mathfrak{t}$ be the dual to the associated linear map. Let η_1, \dots, η_N be **any** (not necessarily normalized) outward conormals.*

- $a(F_i) = F_j$ exactly if $\widehat{a}^*(\eta_i) = \lambda \eta_j$ for some $\lambda > 0$.
- $a(F_i) = F_i$ exactly if $\widehat{a}^*(\eta_i) = \eta_i$.
- $a(F_i) = F_i$ for all i exactly if $a = \text{id}$.

Proof. The first claim is obvious. To prove the second claim, we first note that if $\widehat{a}^*(\eta_i) = \lambda \eta_i$ for some $\lambda > 0$, then

$$\langle \eta_i, \widehat{a}(x) \rangle = \langle \widehat{a}^*(\eta_i), x \rangle = \langle \lambda \eta_i, x \rangle = \lambda \langle \eta_i, x \rangle \quad \forall x \in \Delta.$$

On the other hand, since a_κ is a symmetry of Δ , there exists $c \in \mathbb{R}$ such that $\widehat{a}(\Delta) = \Delta + c$. Therefore

$$\lambda \sup\{\langle \eta_i, x \rangle \mid x \in \Delta\} = \sup\{\langle \eta_i, \widehat{a}(x) \rangle \mid x \in \Delta\} = \sup\{\langle \eta_i, x \rangle \mid x \in \Delta\} + c.$$

Thus if $S := \sup\{\langle \eta_i, x \rangle \mid x \in \Delta\}$ we have $\lambda S = S + c$. Similarly, $\lambda s = s + c$, where $s := \inf\{\langle \eta_i, x \rangle \mid x \in \Delta\}$. Since $S \neq s$, this is impossible unless $\lambda = 1$. Finally, because Δ is bounded the η_i 's span \mathfrak{t}^* , and so the identity is the only symmetry which takes each facet to itself. \square

Example 3.2. Consider the triangle with vertices $(0, 0)$, $(3, 0)$, and $(0, 2)$. Note that this polytope is rational but not smooth. The linear map $\widehat{a}: (x_1, x_2) \mapsto (-x_1 - \frac{3}{2}x_2, \frac{2}{3}x_1)$ is a robust symmetry that gives rise to the symmetry $\widehat{a} + (3, 0)$ of Δ . Since its transpose \widehat{a}^* takes

$(-3, 0)$ to $(0, -2)$, $(0, -2)$ to $(3, 2)$ and $(3, 2)$ to $(-3, 0)$, normalized outward conormals have the form $\lambda(-3, 0)$, $\lambda(0, -2)$, and $\lambda(3, 2)$ for some $\lambda > 0$.

We will also need the following lemma.

Lemma 3.3. *Let F_i and F_j be facets of a smooth polytope $\Delta \subset \mathfrak{t}^*$. If there exists $\xi \in \mathfrak{t}$ such that*

$$\langle \eta_i, \xi \rangle > 0 > \langle \eta'_j, \xi \rangle \quad \text{and otherwise} \quad \langle \eta_k, \xi \rangle = 0 \quad \forall k,$$

then there exists $\xi' \in \ell^$ such that*

$$\langle \eta_i, \xi' \rangle = 1 = -\langle \eta_j, \xi' \rangle \quad \text{and otherwise} \quad \langle \eta_k, \xi' \rangle = 0 \quad \forall k.$$

Proof. By the definition of smooth, the primitive outward conormals at any vertex of Δ form a basis for ℓ . By looking at vertices in $F_i \setminus F_j$ and $F_j \setminus F_i$ one sees that there is a basis of ℓ that contains η_i but not η_j and also a basis that contains η_j but not η_i . Therefore if we choose $\lambda > 0$ so that $\xi' := \lambda \xi$ satisfies $\langle \eta_i, \xi' \rangle = 1$, then ξ' lies in ℓ^* and is primitive; hence we must also have $\langle \eta_j, \xi' \rangle = -1$. \square

We shall say that a linear map $\hat{a} : \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ is a **reflection** if its fixed point set is a hyperplane and $\hat{a} \circ \hat{a} = \text{id}$.

Lemma 3.4. *Let F_i and F_j be distinct facets of a simple polytope $\Delta \subset \mathfrak{t}^*$. The following are equivalent:*

- (i) F_i is equivalent to F_j .
- (ii) *There exists a vector $\xi \in \mathfrak{t}^*$ satisfying*

$$\langle \eta_i, \xi \rangle > 0 > \langle \eta_j, \xi \rangle \quad \text{and otherwise} \quad \langle \eta_k, \xi \rangle = 0 \quad \forall k.$$

- (iii) *There exists a vector $\xi \in \mathfrak{t}^*$ satisfying*

$$\langle \eta_i, \xi \rangle = 1 = -\langle \eta_j, \xi \rangle \quad \text{and otherwise} \quad \langle \eta_k, \xi \rangle = 0 \quad \forall k.$$

- (iv) *There exists a reflection $\hat{a} : \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ which is a robust symmetry of Δ ; for all $\kappa \in \mathcal{C}_\Delta$ the associated symmetry a_κ exchanges F_i and F_j and otherwise takes each facet to itself.*

Proof. It is obvious that (iii) implies (ii) and that (iv) implies (i).

Our first step is to show that (ii) implies (iv). Assume that there exists $\xi \in \mathfrak{t}^*$ such that $\langle \eta_1, \xi \rangle > 0 > \langle \eta_2, \xi \rangle$ and $\langle \eta_k, \xi \rangle = 0$ for all $k > 2$. Let

$$\eta'_1 = \frac{\eta_1}{\langle \eta_1, \xi \rangle} \quad \text{and} \quad \eta'_2 = \frac{\eta_2}{-\langle \eta_2, \xi \rangle};$$

clearly $\langle \eta'_1, \xi \rangle = 1 = -\langle \eta'_2, \xi \rangle$. Define a reflection $\hat{a} : \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ by

$$\hat{a}(x) = x - \langle \eta'_1 - \eta'_2, x \rangle \xi.$$

Given $\kappa \in \mathcal{C}_\kappa$, define an affine map $a_\kappa : \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ by

$$a_\kappa(x) = \hat{a}(x) + (\kappa'_1 - \kappa'_2) \xi, \quad \text{where} \quad \kappa'_1 = \frac{\kappa_1}{\langle \eta_1, \xi \rangle} \quad \text{and} \quad \kappa'_2 = \frac{\kappa_2}{-\langle \eta_2, \xi \rangle}.$$

It is easy to check that a_κ carries $\Delta(\kappa)$ to itself, exchanging F_1 and F_2 and otherwise taking each facet to itself. Moreover, if Δ is smooth, then by Lemma 3.3 we may assume that ξ lies in ℓ^* and that $\langle \eta_1, \xi \rangle = 1 = -\langle \eta_2, \xi \rangle$. Hence, \hat{a} induces an isomorphism of the integer lattice, and so it is a robust symmetry of Δ .

Finally, we will prove that (i) implies (iii). Let $\hat{a}: \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ be a robust symmetry and let $\hat{a}^*: \mathfrak{t} \rightarrow \mathfrak{t}$ denote the dual map. For each $\kappa \in \mathcal{C}_\Delta$, let $a_\kappa: \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ be the associated symmetry of $\Delta(\kappa)$. We can naturally partition the set of facets of Δ into a_κ -orbits; let \mathcal{O} denote the set of such orbits. Assume that each a_κ takes F_1 to F_2 .

Since $a_\kappa(\Delta) = \Delta$, $a_\kappa^k = \text{id}$ for some $k > 0$. Define $p: \mathfrak{t} \rightarrow \mathfrak{t}$ by

$$p = 1 + \hat{a}^* + \cdots + (\hat{a}^*)^{k-1}.$$

We shall find the required vector ξ by studying the properties of this map p .

By our normalization convention (see Remark 1.13), $p(\eta_j) = \frac{k}{|\mathcal{O}|} \sum_{i \in O} \eta_i$ for all j , where O is the orbit containing F_j . Moreover, since Δ is bounded, the η_i span \mathfrak{t} . Hence,

$$(3.1) \quad \ker p = \left\{ \sum \beta_i \eta_i \in \mathfrak{t} \mid \sum_{i \in O} \beta_i = 0 \quad \forall O \in \mathcal{O} \right\}, \quad \text{and}$$

$$(3.2) \quad p(\mathfrak{t}) = \left\{ \sum_{O \in \mathcal{O}} \beta_O \left(\sum_{i \in O} \eta_i \right) \mid \beta_O \in \mathbb{R} \quad \forall O \in \mathcal{O} \right\}.$$

Now fix a nonzero $\beta \in \mathbb{R}^N$ such that $\sum_{i \in O} \beta_i = 0$ for all $O \in \mathcal{O}$, and let $H = \sum \beta_i \eta_i$. Then Proposition 1.18 implies that

$$(3.3) \quad \langle H, c_\Delta(\kappa) \rangle = \sum \beta_i \kappa_i.$$

Since this equality holds for all κ in the open set \mathcal{C}_Δ , this implies that $H \neq 0$. Therefore, $\ker p$ has dimension $N - |\mathcal{O}|$ by (3.1), and so $p(\mathfrak{t}) \subset \mathfrak{t}$ has codimension $N - |\mathcal{O}|$. On the other hand, let $W \subset \mathfrak{t}$ be the subspace spanned by $\eta_1 + \eta_2$ and $\{\eta_k\}_{k>2}$. By (3.2), the fact that F_1 and F_2 are in the same orbit implies that $p(\mathfrak{t})$ is a subspace of W of codimension at most $N - |\mathcal{O}| - 1$. Therefore, W is a proper subspace of \mathfrak{t} . Since the η_i span \mathfrak{t} , this implies that there exists $\xi \in \mathfrak{t}^*$ so that

$$\langle \eta_1, \xi \rangle = 1 = -\langle \eta_2, \xi \rangle \quad \text{and otherwise} \quad \langle \eta_k, \xi \rangle = 0 \quad \forall k.$$

This completes the proof. \square

The above lemma implies that the equivalence relation \sim depends only on the set of conormals of Δ and not on the intersection pattern of its facets. In other words, it depends only on the 1-skeleton of the corresponding fan, not on its higher dimensional simplices.

This lemma (or more precisely, its proof) has the following corollary, which will be useful in §5.

Corollary 3.5. *Let $\Delta \subset \mathfrak{t}^*$ be a simple polytope. Let $\hat{a}: \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ be a linear map so that for all κ' in some neighborhood \mathcal{U} of κ in \mathcal{C}_Δ there exists an affine map a_κ that satisfies conditions (1) and (2) of Definition 1.12. Then \hat{a} is a robust symmetry of Δ .*

Proof. Suppose that the a_κ take F_i to F_j . The above proof that (i) implies (iii) goes through in this situation since all we use is that equation (3.3) holds for all κ in some open set \mathcal{U} . Since (iii) implies (ii) and (ii) implies (iv), there is a robust symmetry that exchanges F_i and F_j but takes every other facet to itself. By composing such maps, we can construct a robust symmetry \widehat{b} so that \widehat{a} and \widehat{b} induce the same permutation on the facets. By Lemma 3.1, this implies that $\widehat{a} = \widehat{b}$. \square

Remark 3.6. Let Δ be a simple polytope. We claim that $H \in \mathfrak{t}$ is inessential on Δ exactly if there exists a robust symmetry \widehat{a} of Δ so that H is in the kernel of the associated projection

$$p = 1 + \widehat{a}^* + \cdots + (\widehat{a}^*)^{k-1},$$

that is, so that H has zero average with respect to the action of \widehat{a}^* . Here, $\widehat{a}^*: \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ is the dual map and $k > 0$ is chosen so that $\widehat{a}^k = \text{id}$.

To see this, note that the lemma above immediately implies that there exists a robust symmetry \widehat{a} of Δ whose set \mathcal{O} of orbits of facets is precisely the set \mathcal{I} of equivalence classes of facets. Moreover, given any robust symmetry \widehat{a} of Δ , all the facets in any orbit $O \in \mathcal{O}$ are equivalent. The claim now follows from the formulas (1.2) and (3.1).

The next lemma gives a very useful criterion for a set of facets to be equivalent.

Lemma 3.7. *Let $\Delta \subset \mathfrak{t}^*$ be a simple polytope. Given a subset $I \subset \{1, \dots, N\}$, $F_i \sim F_j$ for all i and j in I exactly if the subspace $W \subset \mathfrak{t}$ spanned by the outward conormals η_k for $k \notin I$ has codimension $|I| - 1$. Moreover, in this case the linear combination $\sum_{i \in I} c_i \eta_i$ lies in W if and only if $c_i = c_j$ for all i and j .*

Proof. Assume that $F_i \sim F_j$ for all i and j in I . By Lemma 3.4, for each such pair there exists a vector $\xi_{ij} \in \mathfrak{t}^*$ satisfying $\langle \eta_i, \xi_{ij} \rangle = 1 = -\langle \eta_j, \xi_{ij} \rangle$ and otherwise $\langle \eta_k, \xi_{ij} \rangle = 0$ for all k . Fix $i \in I$. Then the vectors ξ_{ij} , $j \in I \setminus \{i\}$, span an $(|I| - 1)$ -dimensional subspace of \mathfrak{t}^* ; let $W \subset \mathfrak{t}$ be its annihilator. By construction, η_k lies in W for all $k \notin I$. On the other hand, since Δ is bounded the positive span of the η_k contains all of \mathfrak{t} . (Recall that the **positive span** of η_1, \dots, η_N is the set $\{\sum a_i \eta_i \mid a_i > 0 \forall i\}$.) In particular, any η_k can be written as a linear combination of the other outward conormals. Hence, the subspace spanned by the η_k for $k \notin I$ has codimension at most $|I| - 1$; so it must be W . Finally, since $\langle \sum_{k \in I} c_k \eta_k, \xi_{ij} \rangle = c_i - c_j$, $\sum_{k \in I} c_k \eta_k$ lies in W exactly if $c_i = c_j$ for all i and j .

Conversely, assume that the plane $W \subset \mathfrak{t}$ spanned by the outward conormals η_k for $k \notin I$ has codimension $|I| - 1$. Hence for any pair $i \neq j$ in I , the subspace spanned by W and the remaining $|I| - 2$ elements in $\{\eta_k, k \in I\}$ is a proper subspace of \mathfrak{t} . Hence there exists $\xi \in \mathfrak{t}^*$ so that $\langle \eta_k, \xi \rangle = 0$ for all k except i and j . Since the positive span of the η_k contains all of \mathfrak{t} , the positive span of the numbers $\langle \eta_k, \xi \rangle$, namely

$$\left\{ \sum a_k \langle \eta_k, \xi \rangle \mid a_k > 0 \right\} = \{a \langle \eta_i, \xi \rangle + b \langle \eta_j, \xi \rangle \mid a, b > 0\},$$

contains all of \mathbb{R} . In particular, after possibly replacing ξ by $-\xi$, we have $\langle \eta_i, \xi \rangle > 0 > \langle \eta_j, \xi \rangle$. By Lemma 3.4, this implies that F_i and F_j are equivalent. \square

Example 3.8. Given simple polytopes $\tilde{\Delta} \subset \tilde{\mathfrak{t}}^*$ and $\hat{\Delta} \subset \hat{\mathfrak{t}}^*$, consider the product $\Delta = \tilde{\Delta} \times \hat{\Delta}$. It follows immediately from Lemma 3.7 above that two facets $\tilde{F}_i \times \hat{\Delta}$ and $\tilde{F}_j \times \hat{\Delta}$ of Δ are equivalent exactly if \tilde{F}_i and \tilde{F}_j are equivalent as facets of $\tilde{\Delta}$. Similarly, two facets $\tilde{\Delta} \times \hat{F}_\ell$ and $\tilde{\Delta} \times \hat{F}_k$ of Δ are equivalent exactly if \hat{F}_ℓ and \hat{F}_k are equivalent as facets of $\hat{\Delta}$. Therefore, $(\tilde{H}, \hat{H}) \in \tilde{\mathfrak{t}} \times \hat{\mathfrak{t}}$ is inessential on Δ exactly if \tilde{H} is inessential on $\tilde{\Delta}$ and \hat{H} is inessential on $\hat{\Delta}$.

The second part of the next result shows that, in the smooth case, if an element of the integer lattice defines an inessential function then this function has rational coefficients.

Lemma 3.9. *Let $\Delta \subset \mathfrak{t}^*$ be a simple polytope; let \mathcal{I} denote the set of equivalence classes of facets. The set of inessential functions is an $(N - |\mathcal{I}|)$ -dimensional subspace of \mathfrak{t} . Moreover, if Δ is smooth then this is a rational subspace, and every inessential $H \in \ell$ can be written*

$$H = \sum \beta_i \eta_i, \quad \text{where } \beta_i \in \mathbb{Q} \ \forall i \text{ and } \sum_{i \in I} \beta_i = 0 \ \forall I \in \mathcal{I}.$$

Proof. Let $H \in \mathfrak{t}$ be inessential. By Definition 1.14 we may write $H = \sum \beta_i \eta_i$, where the β_i satisfy all the above conditions except that they may be in \mathbb{R} not \mathbb{Q} . Fix $i \in I \in \mathcal{I}$. By Lemma 3.4, for all $j \in I \setminus \{i\}$, there exists $\xi_{ij} \in \mathfrak{t}^*$ so that

$$\langle \eta_i, \xi_{ij} \rangle = 1 = -\langle \eta_j, \xi_{ij} \rangle \quad \text{and otherwise} \quad \langle \eta_k, \xi_{ij} \rangle = 0 \ \forall k.$$

Then

$$|I|\beta_i = |I|\beta_i - \sum_{j \in I} \beta_j = \sum_{j \in I} (\beta_i - \beta_j) = \sum_{j \in I \setminus \{i\}} \langle H, \xi_{ij} \rangle.$$

If $H = 0$, this implies immediately that $\beta_i = 0$; this proves the first claim.⁴

On the other hand, if Δ is smooth then, because the $\{\eta_k\}$ form a basis for ℓ , ξ_{ij} lies in the lattice ℓ^* for all $j \in I \setminus \{i\}$. Hence, if $H \in \ell$, then $|I|\beta_i$ is an integer for every $i \in I \in \mathcal{I}$; in particular, $\beta_i \in \mathbb{Q}$. \square

3.2. Bundles and expansions. We will now define two important class of polytopes: bundles and k -fold expansions. Bundles over the simplex Δ_k and k -fold expansions are very similar; in particular, both admit nonconstant inessential functions. Our main result is that these two classes of polytopes are the only ones which admit nonconstant inessential functions.

Definition 3.10. *Let $\tilde{\Delta} = \bigcap_{j=1}^{\tilde{N}} \{x \in \tilde{\mathfrak{t}}^* \mid \langle \tilde{\eta}_j, x \rangle \leq \tilde{\kappa}_j\}$ and $\hat{\Delta} = \bigcap_{i=1}^{\hat{N}} \{y \in \hat{\mathfrak{t}}^* \mid \langle \hat{\eta}_i, y \rangle \leq \hat{\kappa}_i\}$ be simple polytopes. We say that a simple polytope $\Delta \subset \mathfrak{t}^*$ is a **bundle** with **fiber** $\tilde{\Delta}$ over the **base** $\hat{\Delta}$ if there exists a short exact sequence*

$$0 \rightarrow \tilde{\mathfrak{t}} \xrightarrow{\iota} \mathfrak{t} \xrightarrow{\pi} \hat{\mathfrak{t}} \rightarrow 0$$

so that

- Δ is combinatorially equivalent to the product $\tilde{\Delta} \times \hat{\Delta}$. (See Definition 1.7.)
- If $\tilde{\eta}_j'$ denotes the outward conormal to the facet \tilde{F}_j' of Δ which corresponds to $\tilde{F}_j \times \hat{\Delta} \subset \tilde{\Delta} \times \hat{\Delta}$, then $\tilde{\eta}_j' = \iota(\tilde{\eta}_j)$ for all $1 \leq j \leq \tilde{N}$.

⁴ This also follows from Proposition 1.18, as we saw while proving Lemma 3.4.

- If $\hat{\eta}_i'$ denotes the outward conormal to the facet \hat{F}_i' of Δ which corresponds to $\tilde{\Delta} \times \hat{\Delta} \subset \tilde{\Delta} \times \hat{\Delta}$, then $\pi(\hat{\eta}_i') = \hat{\eta}_i$ for all $1 \leq i \leq \tilde{N}$.

The facets $\tilde{F}_1', \dots, \tilde{F}_{\tilde{N}}'$ will be called **fiber facets**, and the facets $\hat{F}_1', \dots, \hat{F}_{\hat{N}}'$ will be called **base facets**.

If Δ is smooth then $\tilde{\Delta}$ and $\hat{\Delta}$ are both smooth. However, the converse does not hold; see Example 1.1.

Remark 3.11. This terminology is justified by the following fact. As we explain in greater detail in §5, there is a manifold M_Δ associated to every smooth polytope Δ . By Remark 5.2, if Δ is a bundle with fiber $\tilde{\Delta}$ over the base $\hat{\Delta}$ in the sense defined above, then M_Δ is a bundle with fiber $M_{\tilde{\Delta}}$ over the base $M_{\hat{\Delta}}$ in the usual sense.

The example below illustrates an important, but slightly confusing, point. The fiber facets are almost never equivalent in any sense to the fiber polytope $\tilde{\Delta}$. Indeed, if the fiber $\tilde{\Delta}$ is a one-simplex, the fiber facets \tilde{F}_j' are analogous to the base polytope $\hat{\Delta}$. In other words we may identify $P(\tilde{F}_j')$ with $\hat{\mathbf{t}}^*$ in such a way as to set up a combinatorial equivalence between \tilde{F}_j' and $\hat{\Delta}$ in which corresponding facets are parallel; cf. the discussion just before Example 1.1. More generally, if the fiber $\tilde{\Delta}$ has dimension k and the base $\hat{\Delta}$ has dimension n , then the non-empty intersection of any k fiber facets is analogous to the base polytope and may be considered as a section of the bundle. Similarly, the non-empty intersection of any n base facets is affine equivalent to the fiber polytope.

Example 3.12. The polytope Y_a defined in Example 1.1 is a Δ_2 bundle over Δ_1 because Δ is combinatorially equivalent to $\Delta_1 \times \Delta_2$ and the conormals $(-1, 0, 0)$, $(0, -1, 0)$, and $(1, 1, 0)$ are linearly dependent. However, it is not a Δ_1 bundle over Δ_2 unless $a_1 = a_2 = 0$.

In contrast, the polytope

$$\{x \in \mathbb{R}^3 \mid x_i \geq 0 \ \forall i, \ x_1 + x_2 \leq \lambda + bx_3, \text{ and } x_3 \leq h\}$$

is a Δ_1 bundle over Δ_2 for all $h > 0$ and $\lambda > \max(0, -bh)$, but is not a Δ_2 bundle over Δ_1 unless $b = 0$. See Figure 3.1.

Definition 3.13. Let $\tilde{\Delta} = \bigcap_{j=1}^{\tilde{N}} \{x \in \tilde{\mathbf{t}}^* \mid \langle \tilde{\eta}_j, x \rangle \leq \tilde{\kappa}_j\}$ be a simple polytope. Given a natural number k , a polytope $\Delta \subset \mathbf{t}^*$ is the **k-fold expansion** of $\tilde{\Delta}$ along the facet \tilde{F}_1 if there is an identification $\mathbf{t} = \tilde{\mathbf{t}} \oplus \mathbb{R}^k$ so that

$$\Delta = \bigcap_{j=2}^{\tilde{N}} \{x \in \mathbf{t}^* \mid \langle \tilde{\eta}_j', x \rangle \leq \tilde{\kappa}_j\} \cap \bigcap_{i=1}^{k+1} \{x \in \mathbf{t}^* \mid \langle \hat{\eta}_i, x \rangle \leq \hat{\kappa}_i\},$$

where $\tilde{\eta}_j' = (\tilde{\eta}_j, 0)$ for $j \geq 2$, $\hat{\eta}_i = (0, -e_i)$ and $\hat{\kappa}_i = 0$ for $1 \leq i \leq k$, $\hat{\eta}_{k+1} = (\tilde{\eta}_1, \sum e_i)$ and $\hat{\kappa}_{k+1} = \tilde{\kappa}_1$. We shall call the facet \tilde{F}_j' of Δ with outward conormal $\tilde{\eta}_j'$ the **fiber-type facet** associated to $\tilde{\mathbf{F}}_j$ and the facet \hat{F}_i with outward conormal $\hat{\eta}_i$ a **base-type facet**.

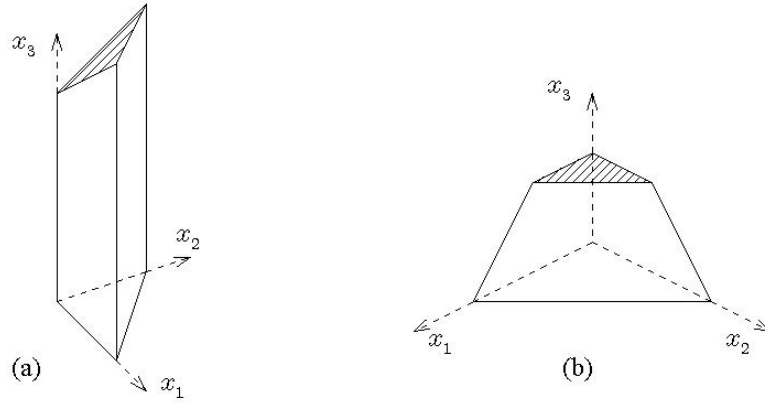


FIGURE 3.1. (a) is a Δ_2 bundle over Δ_1 with a base facet shaded; (b) is a Δ_1 bundle over Δ_2 with a fiber facet shaded. Notice that the fiber facets in (a) contain a common vector that is vertical, while the fiber facets in (b) are parallel.

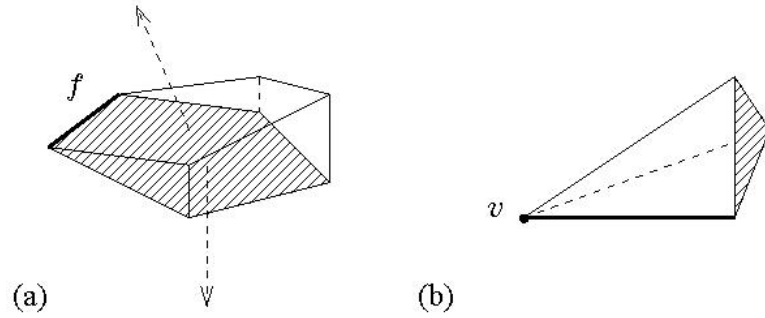


FIGURE 3.2. (a) is the 1-fold expansion of the shaded polygon along $f = \tilde{F}_1$; the two dotted arrows are the conormals to the base-type facets. (b) is the 2-fold expansion of the heavy line at the vertex $v = \tilde{F}_1$. Its one fiber-type facet is dotted.

It is easy to check that Δ is simple, and is smooth exactly if $\tilde{\Delta}$ is. The base-type facets intersect. Indeed, the face $\bigcap_{i=1}^{k+1} \hat{F}_i$ can be identified with \tilde{F}_1 . Similarly, the face $\bigcap_{i \neq n} \hat{F}_i$ can be identified with $\tilde{\Delta}$ for all $1 \leq n \leq k+1$. See Remark 5.4 for a geometric interpretation of 1-fold expansions.

Example 3.14. The k -fold expansion of an n -simplex along one of its facets is an $(n+k)$ -simplex.

Remark 3.15. For $\epsilon > 0$ sufficiently small, the ϵ -blow up Δ' of a polytope Δ along a face $f = F_I = \cap_{i \in I} F_i$ of codimension at least 2 is defined to be the intersection

$$\Delta' = \Delta \cap \{x \in \mathfrak{t}^* \mid \langle \eta'_0, x \rangle \leq \kappa'_0\},$$

where $\eta'_0 := \sum_{i \in I} \eta_i$ and $\kappa'_0 = \sum_{i \in I} \kappa_i - \epsilon$. Thus Δ' is obtained from Δ by cutting out a small neighborhood of f by a suitably angled cut.⁵ Suppose now that Δ is the k -fold expansion of $\tilde{\Delta}$ along \tilde{F}_1 , and consider its blow up along the face $f = \bigcap_{i=1}^{k+1} \tilde{F}_i$. In this case, the conormal to the new facet F'_0 is $\eta'_0 = (\tilde{\eta}_1, 0) \in \mathfrak{t} = \tilde{\mathfrak{t}} \oplus \mathbb{R}^k$. Therefore in the blow up there is a fiber type facet with conormal in $\tilde{\mathfrak{t}} \oplus \{0\}$ corresponding to *each* of the facets of $\tilde{\Delta}$. It is now straightforward to check directly that Δ' is a $\tilde{\Delta}$ bundle over Δ_k and that the base facets are $\tilde{F}_1 \cap \Delta', \dots, \tilde{F}_{k+1} \cap \Delta'$; this justifies our terminology.

The next result is an immediate consequence of Lemma 3.7.

Lemma 3.16. *The base facets of a bundle over a simplex are equivalent, and the base-type facets of a k -fold expansion are equivalent.*

Conversely, the next result shows that if a polytope has equivalent facets, it is either an expansion or a bundle over a simplex.

Proposition 3.17. *Let $\Delta \subset \mathfrak{t}^*$ be a simple (or smooth) polytope. Let $I \in \mathcal{I}$ be an equivalence class of facets and define $I' := I \setminus \{n\}$ for some $n \in I$.*

- (i) *If $F_I = \emptyset$, then Δ is a $F_{I'}$ bundle over $\Delta_{|I|-1}$ with base facets $\{F_i\}_{i \in I}$.*
- (ii) *If $F_I \neq \emptyset$, then Δ is the $(|I| - 1)$ -fold expansion of $F_{I'}$ along $F_I = F_n \cap F_{I'}$ with base-type facets $\{F_i\}_{i \in I}$.*

Proof. Renumber the facets so that $I = \{1, \dots, |I|\}$ and $n = |I|$. By Lemma 3.7, the plane $\tilde{\mathfrak{t}} \subset \mathfrak{t}$ spanned by the η_j for all $j > |I|$ has codimension $|I| - 1$.

Fix any $J \subset \{|I| + 1, \dots, N\}$ so that F_J is not empty and is minimal, in the sense that it has no lower dimensional face $F_{J'} \subsetneq F_J$, where $J' \subset \{|I| + 1, \dots, N\}$. We claim that F_J is affine equivalent to the (standard) $(|I| - 1)$ -simplex with facets $F_i \cap F_J$ for $1 \leq i \leq |I|$; moreover, the η_j for $j \in J$ span $\tilde{\mathfrak{t}}$. To see this, note that the only possible facets of F_J have the form $F_i \cap F_J$, where $1 \leq i \leq |I|$. In particular, F_J has at most $|I|$ facets. On the other hand, the plane $P(F_J)$ contains a translate of the annihilator of $\tilde{\mathfrak{t}}$, which implies that $\dim(F_J) \geq |I| - 1$. The claim follows.

By the paragraph above, $\sum_{1 \leq i \leq |I|} \eta_i$ lies in ${}^6 \tilde{\mathfrak{t}}$ and $\{\eta_i\}_{1 \leq i < |I|}$ descends to a basis for the quotient of \mathfrak{t} by $\tilde{\mathfrak{t}}$. Hence, we can identify \mathfrak{t} with $\tilde{\mathfrak{t}} \oplus \mathbb{R}^{|I|-1}$ so that $\eta_i = (0, -e_i)$ for all $1 \leq i < |I|$ and $\eta_{|I|} = (\alpha, \sum_{i=1}^{|I|-1} e_i)$ for some $\alpha \in \tilde{\mathfrak{t}}$. Moreover, let $J \subset \{|I| + 1, \dots, N\}$ be any subset so that the face F_J is not empty. By the paragraph above, the intersection $F_J \cap F_{\hat{I}}$ is also not empty for any proper subset $\hat{I} \subsetneq I$. Hence, if $F_I = \emptyset$, then Δ is combinatorially equivalent to the product $F_{I'} \times \Delta_{|I|-1}$, and so Δ is a $F_{I'}$ bundle over $\Delta_{|I|-1}$. On the other

⁵ We give more details of this construction in Part II.

⁶ This holds even in the nonsmooth case by our conventions for normalizing the η_i ; see Remark 1.13 and Example 1.15.

hand, if $F_I \neq \emptyset$ then the outward conormal to the facet $F_I = F_{|I|} \cap F_{I'}$ of the polytope $F_{I'}$ is the image of $\eta_{|I|}$ in the quotient of \mathfrak{t} by the span of $\{\eta_i\}_{i \in I'}$. This is $\alpha \in \tilde{\mathfrak{t}}$, as required. \square

Corollary 3.18. *Let Δ be a simple polytope such that all its facets are equivalent. Then $\Delta = \Delta_k$.*

Proof. In this case there is just one equivalence class of facets I . Since Δ is bounded, $F_I = \emptyset$. Now apply part (i) of the above proposition. \square

3.3. Subtracting inessential functions. In this subsection, we show that it is often possible to use Proposition 1.18 to simplify a mass linear function by subtracting an inessential function; the resulting function will have fewer asymmetric facets.

Lemma 3.19. *Let $H \in \mathfrak{t}$ be a mass linear function on a simple polytope $\Delta \subset \mathfrak{t}^*$. If F_1, \dots, F_m are equivalent facets, there exists an inessential function $H' \in \mathfrak{t}$ so that the mass linear function $\tilde{H} = H - H'$ has the following properties:*

- For all $i < m$, the facet F_i is \tilde{H} -symmetric.
- For all $i > m$, the facet F_i is \tilde{H} -symmetric iff it is H -symmetric.

Proof. Since H is mass linear, $\langle H, c_\Delta \rangle = \sum_{i=1}^N \beta_i \kappa_i$ for some $\beta_i \in \mathbb{R}$. Define $\alpha_i = \beta_i$ for all $1 \leq i < m$ and define $\alpha_m = -\sum_{i=1}^{m-1} \beta_i$. Then $\sum_{i=1}^m \alpha_i = 0$, and so by definition $H' = \sum_{i=1}^m \alpha_i \eta_i$ is inessential. By Proposition 1.18, $\langle H', c_\Delta \rangle = \sum_{i=1}^m \alpha_i \kappa_i$. Therefore, if $\tilde{H} = H - H'$ then

$$\langle \tilde{H}, c_\Delta \rangle = \left(\sum_{i=1}^m \beta_i \right) \kappa_m + \sum_{i=m+1}^N \beta_i \kappa_i,$$

as required. \square

Corollary 3.20. *Let $H \in \mathfrak{t}$ be a mass linear function on a simple polytope $\Delta \subset \mathfrak{t}^*$. If the asymmetric facets are all equivalent, then H is inessential.*

Proof. By Lemma 3.19, there exists an inessential function $H' \in \mathfrak{t}$ so that the mass linear function $\tilde{H} = H - H'$ has at most one asymmetric facet. By Lemma 2.13, this implies that $\tilde{H} = 0$. \square

This method of simplification works particularly well when the equivalent facets do not intersect, that is, when our polytope is a bundle over a simplex. To prove this, we will need the following lemma.

Lemma 3.21. *Let $H \in \mathfrak{t}$ be a mass linear function on a simple polytope $\Delta \subset \mathfrak{t}^*$. Assume that Δ is a $\tilde{\Delta}$ bundle over the simplex Δ_k , and let $\tilde{\mathfrak{t}} \subset \mathfrak{t}$ be the subspace spanned by the conormals to the fiber facets. Then H lies in $\tilde{\mathfrak{t}}$ exactly if the base facets are symmetric.*

Proof. Let $\tilde{\eta}_1, \dots, \tilde{\eta}_{\tilde{N}}$ be the outward conormals to the fiber facets, and let $\hat{\eta}_1, \dots, \hat{\eta}_{k+1}$ be the outward conormals to the base facets. Let $\tilde{\kappa}_1, \dots, \tilde{\kappa}_{\tilde{N}}$ and $\hat{\kappa}_1, \dots, \hat{\kappa}_{k+1}$ be the associated support numbers.

If the base facets are symmetric then H lies in $\tilde{\mathfrak{t}}$ by Lemma 2.6. To prove the converse, assume that H lies in $\tilde{\mathfrak{t}}$. Because $\tilde{\Delta}$ is bounded, the positive span of the $\tilde{\eta}_i$ contains all of $\tilde{\mathfrak{t}}$. In particular, $H = \sum \beta_i \tilde{\eta}_i$, where $\beta_i > 0$ for all i . Because $\langle \tilde{\eta}_j, x \rangle \leq \tilde{\kappa}_j$ for all $x \in \Delta$, we have $\langle \tilde{\eta}_j, c_\Delta \rangle \leq \tilde{\kappa}_j$ for all j . Hence $\langle H, c_\Delta \rangle \leq \sum \beta_j \tilde{\kappa}_j$. The number $\sum \beta_j \tilde{\kappa}_j$ does not depend on the $\hat{\kappa}_i$. On the other hand, because the base is a simplex, we can make any $\hat{\kappa}_i$ arbitrarily large without changing the combinatorics of the polytope Δ . Since $\langle H, c_\Delta \rangle$ is a linear function on \mathcal{C}_Δ , this implies that it cannot depend positively on any $\hat{\kappa}_i$. The same argument for $-H$ shows that $\langle H, c_\Delta \rangle$ cannot depend negatively on any $\hat{\kappa}_i$. \square

Proposition 3.22. *Let $H \in \mathfrak{t}$ be a mass linear function on a simple polytope $\Delta \subset \mathfrak{t}^*$ which is a bundle over the simplex Δ_k . There exists an inessential function $H' \in \mathfrak{t}$ so that the mass linear function $\tilde{H} = H - H'$ has the following properties:*

- The base facets are \tilde{H} -symmetric
- A fiber facet is \tilde{H} -symmetric iff it is H -symmetric.

Proof. Let $\hat{\eta}_1, \dots, \hat{\eta}_{k+1}$ denote the outward conormals to the base facets. By Lemma 3.7 and the definition of a bundle, the base facets are equivalent. Moreover, we may decompose the vector $H \in \mathfrak{t}$ uniquely as

$$H = \tilde{H} + \sum_{i=1}^{k+1} \alpha_i \hat{\eta}_i, \quad \text{where} \quad \sum \alpha_i = 0 \quad \text{and} \quad \tilde{H} \in \tilde{\mathfrak{t}},$$

where $\tilde{\mathfrak{t}} \subset \mathfrak{t}$ is the subspace spanned by the conormals of the fiber facets. The function $H' = \sum_i \alpha_i \hat{\eta}_i$ is inessential by definition. Hence, by Proposition 1.18, H' is mass linear and $\langle H', c_\Delta \rangle$ depends only on the support numbers of the base facets. Thus \tilde{H} is mass linear and the fiber facets are \tilde{H} -symmetric exactly if they are H -symmetric. The base facets are \tilde{H} -symmetric by Lemma 3.21. \square

3.4. Flat facets. In this subsection, we consider flat facets. The main result (Proposition 3.25) allows us to reduce our discussion of the structure of mass linear pairs (Δ, H) to the case that every asymmetric facet is pervasive.

If Δ is a bundle over Δ_1 then the base facets are flat and do not intersect. The lemma below proves the converse.

Lemma 3.23. *Let $\Delta \subset \mathfrak{t}^*$ be a simple polytope. Let F_1 and F_2 be flat facets of Δ which do not intersect. Then Δ is a bundle over the simplex Δ_1 with base facets F_1 and F_2 .*

Proof. By definition, since F_1 is flat there exists a nonzero $\xi_1 \in \mathfrak{t}^*$ so that $\langle \eta_j, \xi_1 \rangle = 0$ for every facet F_j (other than F_1 itself) which meets F_1 . Since F_1 is bounded, these η_j span \mathfrak{t}/η_1 ; hence $\langle \eta_1, \xi_1 \rangle \neq 0$ and so we can renormalize ξ_1 so that $\langle \eta_1, \xi_1 \rangle = 1$. Similarly, there exists $\xi_2 \in \mathfrak{t}^*$ so that $\langle \eta_2, \xi_2 \rangle = 1$ but $\langle \eta_j, \xi_2 \rangle = 0$ for every (other) facet F_j which meets F_2 .

We claim that

$$(3.4) \quad \langle \eta_k, \xi_1 \rangle \leq 0 \quad \forall k \neq 1 \quad \text{and} \quad \langle \eta_k, \xi_2 \rangle \leq 0 \quad \forall k \neq 2.$$

To see this, pick a point x which lies in some F_k but not in F_1 . By definition $\langle \eta_k, x \rangle = \kappa_k$, $\langle \eta_1, x \rangle < \kappa_1$, and $\langle \eta_i, x \rangle \leq \kappa_i$ for all i . Define $y := x + (\kappa_1 - \langle \eta_1, x \rangle) \xi_1$. An easy calculation

shows that $\langle \eta_1, y \rangle = \kappa_1$ and $\langle \eta_k, y \rangle = \kappa_k + (\kappa_1 - \langle \eta_1, x \rangle) \langle \eta_k, \xi_1 \rangle$. If $F_i \neq F_1$ is any facet which meets F_1 then $\langle \eta_i, \xi_1 \rangle = 0$, and hence $\langle \eta_i, y \rangle = \langle \eta_i, x \rangle \leq \kappa_i$. Therefore, $y \in F_1 \subset \Delta$. In particular, $\langle \eta_k, y \rangle \leq \kappa_k$ for all k . Since $\langle \eta_1, x \rangle < \kappa_1$, this implies that $\langle \eta_k, \xi_1 \rangle \leq 0$, as required.

Next, we claim that ξ_2 is a multiple of ξ_1 . To see this, let J denote the set of $j \in \{2, 3, \dots, N\}$ so that F_j meets F_1 . On the one hand, since $\langle \eta_j, \xi_1 \rangle = 0$ for all $j \in J$ and since F_1 is bounded, the positive span of the η_j for $j \in J$ contains ξ_1° , the annihilator of ξ_1 . On the other hand, since $F_1 \cap F_2 = \emptyset$, 2 is not in J . Hence (3.4) implies that $\langle \eta_j, \xi_2 \rangle \leq 0$ for all $j \in J$. Therefore, $\langle \alpha, \xi_2 \rangle \leq 0$ for all $\alpha \in \xi_1^\circ$. Since ξ_1° is a linear subspace, this is possible only if $\langle \alpha, \xi_2 \rangle = 0$ for all $\alpha \in \xi_1^\circ$. Thus ξ_2 is a multiple of ξ_1 as claimed.

Since $\langle \eta_2, \xi_2 \rangle = 1$ and $\langle \eta_1, \xi_2 \rangle \leq 0$ by (3.4), ξ_2 cannot be a positive multiple of ξ_1 ; so it must be a negative multiple. Hence, for all k except 1 and 2, we have $\langle \eta_k, \xi_1 \rangle \leq 0$ and $\langle \eta_k, \xi_1 \rangle \geq 0$, that is, $\langle \eta_k, \xi_1 \rangle = 0$.

To see that F_1 and F_2 are the base facets of a bundle over Δ_1 it now suffices to check that Δ is combinatorially equivalent to the product $F_1 \times \Delta_1$. Equivalently, we must check that every non-empty face F_J , where $J \subset \{3, \dots, N\}$, meets both F_1 and F_2 . To see this, pick $x \in F_J$ and define $y := x + (\kappa_1 - \langle \eta_1, x \rangle) \xi_1$ as before. Since $\langle \eta_1, \xi_1 \rangle = 1$, $\langle \eta_2, \xi_1 \rangle < 0$, and otherwise $\langle \eta_j, \xi_1 \rangle = 0$, $y \in F_1 \cap F_J$. A similar argument shows that $F_2 \cap F_J \neq \emptyset$. \square

Corollary 3.24. *Let $H \in \mathfrak{t}$ be a mass linear function on a simple polytope $\Delta \subset \mathfrak{t}^*$. If F is an asymmetric facet which is not pervasive, then Δ is an F bundle over Δ_1 .*

Proof. Let G be any facet which is disjoint from F . By Lemma 2.8, G is also asymmetric. Hence, Proposition 2.11 implies that F and G are flat. By Lemma 3.23, this implies that Δ is an F bundle over Δ_1 . \square

Here is the main result of this section.

Proposition 3.25. *Let $H \in \mathfrak{t}$ be a mass linear function on a simple polytope $\Delta \subset \mathfrak{t}^*$. There exists an inessential function $H' \in \mathfrak{t}$ so that the mass linear function $\tilde{H} = H - H'$ has the following property: every \tilde{H} -asymmetric facet is pervasive.*

Proof. We will argue this by induction on the number of asymmetric facets which are not pervasive. Let F be an H -asymmetric facet which is not pervasive. By Corollary 3.24, Δ is an F bundle over Δ_1 . By Proposition 3.22 there exists an inessential function $H' \in \mathfrak{t}$ so that the mass linear function $\tilde{H} = H - H'$ has the following properties: the base facets (including F) are \tilde{H} -symmetric, and the fiber facets are \tilde{H} -symmetric exactly if they are H -symmetric. Hence, there are fewer \tilde{H} -asymmetric facets which are not pervasive. Since the sum of inessential functions is inessential, this completes the proof. \square

3.5. Polytopes such that all linear functions are mass linear. This subsection is devoted to a proof of Theorem 1.20. First, note that every inessential function is mass linear by Proposition 1.18. Next, note that by Example 1.16, if $\Delta \subset \mathfrak{t}^*$ is a product of simplices, then every $H \in \mathfrak{t}$ is an inessential mass linear function on Δ , and $\bigcap_{i \in I} F_i = \emptyset$ for every equivalence class of facets I . Thus (iii) \implies (ii) \implies (i) and (iii) \implies (iv).

To complete the proof, we will repeatedly need the following elementary lemmas.

Lemma 3.26. *Let $\Delta \subset \mathfrak{t}^*$ be a simple polytope. Each facet F_i is η_i -asymmetric.*

Proof. Assume that F_i is η_i -symmetric. By Lemma 2.7, this implies that $\langle \eta_i, c_{F_i} \rangle = \langle \eta_i, c_\Delta \rangle$. Since $\langle \eta_i, c_{F_i} \rangle = \kappa_i$, this contradicts the assumption that F_i is η_i -symmetric. \square

Lemma 3.27. *Let F_i be a pervasive facet of a simple polytope $\Delta \subset \mathfrak{t}^*$, and let $j, k \neq i$. Then $F_{ji} = F_j \cap F_i$ and $F_{ki} = F_k \cap F_i$ are equivalent as facets of F_i exactly if F_j and F_k are equivalent as facets of Δ .*

Proof. By Lemma 3.7, $F_j \sim F_k$ exactly if there is a vector $\xi \in \mathfrak{t}^*$ that lies in the annihilator η_ℓ° of the conormal η_ℓ for all $\ell \neq j, k$.

Since F_i is pervasive, $F_{\ell i}$ is a facet of F_i for every $\ell \neq i$. Moreover, $P(F_i)$ is naturally affine isomorphic to η_i° . Therefore, F_{ji} and F_{ki} are facets of F_i , and they are equivalent facets exactly if there exists $\xi \in \eta_i^\circ \subset \mathfrak{t}^*$ that lies in the annihilator of η_ℓ° for all $\ell \neq i, j, k$.

But these two conditions are clearly equivalent. \square

Proof that (i) \implies (iii) in Theorem 1.20.

Let $\Delta \subset \mathfrak{t}^*$ be a simple polytope; let \mathcal{I} denote the set of equivalence classes of facets. Assume that every $H \in \mathfrak{t}$ is mass linear on Δ . We will prove that Δ is a product of simplices. The proof is divided into a sequence of steps.

Claim I: *If $\dim \Delta \leq 2$, then Δ is a product of simplices.*

The claim is trivial in dimension 1. So assume that $\dim \Delta = 2$ and that $\Delta \neq \Delta_2$. Since Δ has more than three edges, no edge is pervasive. Moreover, each outward conormal η_i is mass linear by assumption, and each edge F_i is η_i -asymmetric by Lemma 3.26. Therefore, Proposition 2.11 implies that each edge F_i of Δ is flat. But this is possible only if Δ is the product $\Delta_1 \times \Delta_1$.

We will now assume that $\dim \Delta \geq 3$ and that (i) \implies (iii) for all lower dimensional polytopes.

Claim II: *Each proper face of Δ is a product of simplices.*

Since we have assumed that (i) \implies (iii) holds for lower dimensional polytopes, it is enough to prove every facet F_i of Δ satisfies condition (i); that is, we must show that every H' in $\mathfrak{t}/\langle \eta_i \rangle$ – which is naturally the dual space to $P(F_i)$ – is mass linear on F_i . To see this, let \mathcal{H}_i denote the set of $H \in \mathfrak{t}$ so that the facet F_i is H -symmetric. By Lemma 3.26, $\eta_i \notin \mathcal{H}_i$. On the other hand, since every $H \in \mathfrak{t}$ is mass linear on Δ , there is $\lambda \in \mathbb{R}$ such that the coefficient of κ_i in $H - \lambda \eta_i$ is zero, that is, $H - \lambda \eta_i \in \mathcal{H}_i$. Hence $\mathcal{H}_i \subset \mathfrak{t}$ is a linear subspace of codimension one and so the natural projection from \mathcal{H}_i to $\mathfrak{t}/\langle \eta_i \rangle$ is surjective. Finally, by Proposition 2.9, each $H \in \mathcal{H}_i$ restricts to a mass linear function on F_i .

Claim III: *If F_1 is not pervasive, Δ is a bundle over the 1-simplex Δ_1 .*

Since $\eta_1 \in \mathfrak{t}$ is a mass linear function on Δ and F_1 is η_1 -asymmetric, Δ is a bundle over Δ_1 by Corollary 3.24.

Claim IV: *If every facet of Δ is pervasive, $\bigcap_{i \in I} F_i = \emptyset$ for all $I \in \mathcal{I}$.*

Let $\{F_1, \dots, F_{k+1}\}$ be an equivalence class of facets. If Δ has no other facets, then the conclusion holds because Δ is bounded. (Alternatively, $\Delta = \Delta_k$ by Corollary 3.18.) Otherwise, fix $j > k + 1$. Since F_j is pervasive, $\{F_{ij}\}_{i=1}^{k+1}$ is a nonempty equivalence class of facets of F_j by Lemma 3.27. Moreover, by Claim II, F_j is the product of simplices. Hence, by Example 1.16, $\bigcap_{i=1}^{k+1} F_{ij} = \emptyset$. In particular, $k > 0$. Therefore, a nearly identical argument shows that $\bigcap_{i=2}^{k+1} F_{i1} = \emptyset$. But $\bigcap_{i=2}^{k+1} F_{i1} = \bigcap_{i=1}^{k+1} F_i$.

Claim V: Δ is a bundle over the k -simplex Δ_k , where $k \neq \dim \Delta - 1$.

If some facet of Δ is not pervasive, then Δ is a bundle over Δ_1 by Claim III. Since $\dim \Delta \geq 3$, $\dim \Delta - 1 \neq 1$. So assume instead that every facet is pervasive. By Claim IV, $\bigcap_{i \in I} F_i = \emptyset$ for all $I \in \mathcal{I}$. Therefore, applying Proposition 3.17, we see that Δ is a bundle over the k -simplex Δ_k . Finally, if $k = \dim \Delta - 1$ then the fiber is 1-dimensional; hence it is Δ_1 . This contradicts the claim that every facet is pervasive.

Claim VI: Δ is a product of simplices.

By Claim V, Δ is a bundle over the k -simplex Δ_k , where $k \neq \dim \Delta - 1$. Label the base facets $\{F_1, \dots, F_{k+1}\}$. If $k = \dim \Delta$ we are done, so assume that $k \leq \dim \Delta - 2$.

Since $k + 1 < \dim \Delta$ and Δ is bounded, there exists $i > k + 1$ so that η_i does not lie in the span of $\eta_1, \dots, \eta_{k+1}$. Since Δ is a bundle over Δ_k with base facets F_1, \dots, F_{k+1} , the facet F_i is also a bundle over Δ_k with base facets $F_1 \cap F_i, \dots, F_{k+1} \cap F_i$. Moreover, identifying the dual space to $P(F_i)$ with $\mathfrak{t}/\langle \eta_i \rangle$ as in Claim II, we see that the projections of $\eta_1, \dots, \eta_{k+1}$ to $\mathfrak{t}/\langle \eta_i \rangle$ are the conormals to the base facets $F_1 \cap F_i, \dots, F_{k+1} \cap F_i$ of F_i . But the bundle is trivial by Claim II. Hence these conormals are linearly dependent in the quotient space. Since η_i does not lie in the span of $\eta_1, \dots, \eta_{k+1}$, this implies that $\eta_1, \dots, \eta_{k+1}$ are themselves linearly dependent, that is, Δ is a trivial bundle over Δ_k . Finally, the fiber of this bundle is naturally isomorphic to the face $\bigcap_{i=2}^{k+1} F_i$. By Claim II, this face is a product of simplices. \square

Proof that (iv) \implies (iii) in Theorem 1.20.

Let $\Delta \subset \mathfrak{t}^*$ be a simple polytope; let \mathcal{I} denote the set of equivalence classes of facets. Assume that $\bigcap_{i \in I} F_i = \emptyset$ for all $I \in \mathcal{I}$; in particular, $|I| > 1$. We will prove that Δ is a product of simplices. As before, we divide the proof into a sequence of steps.

Claim I: If $\dim \Delta \leq 2$, then Δ is a product of simplices.

If $\dim \Delta = 2$ and there is an equivalence class with more than two elements, or if there is an equivalence class with exactly two edges which meet, then Proposition 3.17 implies that Δ is Δ_2 . Otherwise Δ is a Δ_1 -bundle over Δ_1 such that its two fiber facets are equivalent. By Lemma 3.7, this implies that Δ is a product.

We will now assume that (iv) \implies (iii) for all lower dimensional polytopes.

Claim II: Every proper face of Δ is a product of simplices.

Since we have assumed that (iv) \implies (iii) holds for lower dimensional polytopes, it is enough to prove that every facet F_i satisfies condition (iv); in other words, we must show that $\bigcap_{i \in I} F_{i1} = \emptyset$ for every equivalence class $\{F_{i1}\}_{i \in I}$ of facets of F_1 . Even if F_1 is not

pervasive, it is clear from the definitions that if F_i and F_j are equivalent facets of Δ that intersect F_1 then F_{i1} and F_{j1} are equivalent facets of Δ . Therefore, for each equivalence class I of facets of F_1 , the set $\{F_1\} \cup \{F_i\}_{i \in I}$ contains an equivalence class of facets of Δ . By assumption, this implies that $F_1 \cap \bigcap_{i \in I} F_i = \emptyset$. But $F_1 \cap \bigcap_{i \in I} F_i = \bigcap_{i \in I} F_{i1}$.

Claim III: *If F_1 is not pervasive, Δ is a bundle over the 1-simplex Δ_1 .*

Let $I \in \mathcal{I}$ be the equivalence class of facets which contains F_1 . By assumption, $\bigcap_{i \in I} F_i = \emptyset$. By Proposition 3.17, this is only possible if Δ is a bundle over Δ_1 .

The rest of the proof now follows exactly as in the previous case, although in fact Claim IV is true by assumption. \square

4. POLYTOPES IN 2 AND 3 DIMENSIONS

In this section, we give a complete classification of polygons and smooth 3-dimensional polytopes which admit mass linear functions. Here, a **(convex) polygon** is a simple 2-dimensional polytope.

4.1. Polygons. In this subsection, we describe mass linear functions on polygons, showing that they are all inessential.

We shall need the following lemma; since every edge has only two facets, it is an immediate consequence of Lemmas 2.8 and 2.13.

Lemma 4.1. *Fix a nonzero $H \in \mathfrak{t}$ and a simple polytope $\Delta \subset \mathfrak{t}^*$. If Δ has a symmetric edge then it has exactly two asymmetric facets.*

Here is our main result; it is an elaboration of Theorem 1.3.

Proposition 4.2. *Let $H \in \mathfrak{t}$ be a nonzero mass linear function on a polygon $\Delta \subset \mathfrak{t}^*$. Then one of the following statements holds:*

- Δ is a triangle; at most one edge is symmetric.
- Δ is a Δ_1 bundle over Δ_1 ; the base facets are the asymmetric edges.
- Δ is the product $\Delta_1 \times \Delta_1$; each edge is asymmetric.

In any case, H is inessential.

Proof. Assume that Δ has a symmetric edge. By Lemma 4.1, Δ has two asymmetric edges. Moreover, by Lemma 2.8, every symmetric edge must meet both asymmetric edges; in particular, there are at most two symmetric edges. If there is one symmetric edge, Δ is a triangle. If there are two symmetric edges, then the asymmetric edges are not pervasive and so by Proposition 2.11 they are flat. This implies that Δ is a Δ_1 bundle over Δ_1 , and that the fiber facets are symmetric. In either case, the asymmetric facets are equivalent. Hence, H is inessential by Corollary 3.20.

So assume that every edge is asymmetric. If Δ has three edges, it is a triangle. Otherwise, since none of the edges are pervasive Proposition 2.11 implies that they are all flat. By definition, this is impossible unless Δ is the product $\Delta_1 \times \Delta_1$. Moreover, all the facets of a triangle are equivalent, and the two pairs of opposite facets of $\Delta_1 \times \Delta_1$ are equivalent. It is easy to check that in either case every $H' \in \mathfrak{t}$ is inessential. \square

The next corollary will be useful in Part II.

Corollary 4.3. *Let $H \in \mathfrak{t}$ be a mass linear function on a polygon $\Delta \subset \mathfrak{t}^*$. If two edges F_i and F_j do not intersect then $\gamma_i + \gamma_j = 0$, where γ_k is the coefficient of the support number of F_k in the linear function $\langle H, c_\Delta \rangle$.*

Proof. If $H \neq 0$, then the lemma above implies that H is inessential and F_i and F_j are opposite edges of a quadrilateral. Hence while they may (or may not) be equivalent to each other, neither is equivalent to any other edge. Therefore Proposition 1.18 implies that either $\gamma_i = \gamma_j = 0$ (if $F_i \not\sim F_j$) or $\gamma_i + \gamma_j = 0$. \square

4.2. Polytopes with two asymmetric facets. We now show that any mass linear function on a simple polytope with at most two asymmetric facets is inessential. In this paper, we need this result for the 3-dimensional case, but it is valid in all dimensions.

Proposition 4.4. *Let $H \in \mathfrak{t}$ be a mass linear function on a simple polytope $\Delta \subset \mathfrak{t}^*$ with exactly two asymmetric facets. Then the asymmetric facets are equivalent.*

Proof. Let F_1 and F_2 be the asymmetric facets.

First consider a 2-dimensional symmetric face Y with two symmetric edges e and e' . By Proposition 2.9, Y has exactly two asymmetric edges $F_1 \cap Y$ and $F_2 \cap Y$, and the restriction of H to Y is mass linear. Therefore by Proposition 4.2, Y is a Δ_1 bundle over Δ_1 and e and e' are fiber facets. Hence they are parallel.

More generally, define a graph Γ as follows: let $V := V(\Gamma)$ be the set of vertices in $F_1 \setminus (F_1 \cap F_2)$ and let $E := E(\Gamma)$ be the set of edges which join such vertices. By Lemma 2.8 each symmetric edge intersects F_1 and F_2 at two different points; hence, it must intersect F_1 in a point of V . On the other hand, given any $v \in V$, since F_1 is the only asymmetric facet through v and since Δ is simple, v must lie on a unique symmetric edge. Hence, intersection with F_1 induces a one-to-one correspondence between the set of symmetric edges and V . By a similar argument, intersection with F_1 also a one-to-one correspondence between the set of 2-dimensional symmetric faces Y with two symmetric edges and E . (Here, we use the fact proved above that the two symmetric edges in Y are parallel and so cannot meet.) Hence, if e and e' are symmetric edges so that $e \cap F_1$ and $e' \cap F_1$ lie in the same component of Γ , then e and e' are parallel. Since Γ is clearly connected, this implies that all symmetric edges are parallel.

Every symmetric facet G contains a symmetric face g which is minimal in the sense that it has no symmetric faces. By Proposition 2.9, g is a polytope with exactly two facets $F_1 \cap g$ and $F_2 \cap g$. This is only possible if g is a symmetric edge. Therefore, the conormals to the symmetric facets lie in a codimension 1 subspace. By Lemma 3.7, this implies that the asymmetric facets are equivalent. \square

This proposition has the following important consequence.

Corollary 4.5. *Let $H \in \mathfrak{t}$ be a mass linear function on a simple polytope $\Delta \subset \mathfrak{t}^*$ with exactly two asymmetric facets F_1 and F_2 . Then H is inessential, and exactly one of the following occurs.*

- Δ is a F_1 bundle over Δ_1 ; the base facets are the asymmetric facets, or
- Δ is the expansion of F_1 along F_{12} ; the base-type facets are the asymmetric facets.

Proof. By Proposition 4.4, F_1 and F_2 are equivalent. Thus the claim follows immediately from Corollary 3.20 and Proposition 3.17. \square

4.3. Mass linear functions on the polytopes defined in Example 1.1. In this subsection, we calculate all mass linear functions on the polytopes defined in Example 1.1. In particular, we show that these functions are not all inessential. Let's recall this example. Given $a = (a_1, a_2) \in \mathbb{R}^2$, define

$$\eta_1 = -e_1, \eta_2 = -e_2, \eta_3 = e_1 + e_2, \eta_4 = -e_3, \eta_5 = e_3 + a_1 e_1 + a_2 e_2, \text{ and}$$

$$\mathcal{C}_a = \left\{ \kappa \in \mathbb{R}^5 \left| \sum_{i=1}^3 \kappa_i > 0 \text{ and } \kappa_4 + \kappa_5 > -a_1 \kappa_1 - a_2 \kappa_2 + \max(0, a_1, a_2) \sum_{i=1}^3 \kappa_i \right. \right\}.$$

Given $\kappa \in \mathcal{C}_a$, let

$$Y = Y(\kappa) = \bigcap_{i=1}^5 \{x \in (\mathbb{R}^3)^* \mid \langle \eta_i, x \rangle \leq \kappa_i\}.$$

Here is the main result of this section.

Proposition 4.6. *Let Y be the polytope defined in Example 1.1. Then $H \in \mathfrak{t}$ is mass linear function on Y exactly if*

$$H = \sum_{i=1}^5 \gamma_i \eta_i, \text{ where } \gamma_1 + \gamma_2 + \gamma_3 = 0, \gamma_4 + \gamma_5 = 0, \text{ and } a_1 \gamma_1 + a_2 \gamma_2 = 0.$$

In this case $\langle H, c_Y \rangle = \sum_{i=1}^5 \gamma_i \kappa_i$; moreover, H is inessential exactly if at least one of the following conditions holds: $a_1 a_2 (a_1 - a_2) = 0$, or $\gamma_1 = \gamma_2 = \gamma_3 = 0$.

Remark 4.7. In particular, all vectors $H \in \mathfrak{t}$ are mass linear on Y (and are inessential) exactly if $a_1 = a_2 = 0$ and Y is the product $\Delta_1 \times \Delta_2$. This is consistent with Theorem 1.20. Otherwise, Δ admits a 2-dimensional family of mass linear functions. If $a_1 a_2 (a_1 - a_2) = 0$, these functions are all inessential. Otherwise, there is a 1-dimensional family of inessential functions; the rest are essential.

The proof of this proposition rests mainly on the following direct calculation.

Lemma 4.8. *Let Y be the polytope defined in Example 1.1, and let $H = \gamma_1 \eta_1 + \gamma_2 \eta_2 + \gamma_3 \eta_3$, where $\gamma_1 + \gamma_2 + \gamma_3 = 0$. Then H is mass linear on Y if and only if*

$$\gamma_1 a_1 + \gamma_2 a_2 = 0; \quad \text{in this case} \quad \langle H, c_Y \rangle = \sum_{i=1}^3 \gamma_i \kappa_i.$$

Proof. As a first step, fix $\kappa_1 = \kappa_2 = \kappa_4 = 0$, and let $\kappa_3 = \lambda$ and $\kappa_5 = h$. Let $\Delta_2^\lambda \subset \mathbb{R}^2$ denote the 2-simplex given by the inequalities

$$x_1 \geq 0, \quad x_2 \geq 0, \quad \text{and} \quad x_1 + x_2 \leq \lambda.$$

An elementary calculation shows that for any non-negative integers i_1 and i_2

$$\int_{\Delta_2^\lambda} x_1^{i_1} x_2^{i_2} = \frac{i_1! i_2! \lambda^{I+2}}{(I+2)!},$$

where $I = i_1 + i_2$ and where by convention $0! = 1$. Further, both here and below we integrate with respect to the standard measure $dx_1 dx_2$ on \mathbb{R}^2 .

Since Y is a Δ_2^λ bundle over Δ_1 , Y has volume

$$V = \int_{\Delta_2^\lambda} (h - a_1 x_1 - a_2 x_2) = \frac{3h\lambda^2 - (a_1 + a_2)\lambda^3}{3!}.$$

For $j \neq 3$, the moment μ_j of Y along the x_j axis is

$$\mu_j = \int_{\Delta_2^\lambda} ((h - a_1 x_1 - a_2 x_2) x_j) = \frac{4h\lambda^3 - (a_j + a_1 + a_2)\lambda^4}{4!}.$$

Let $c_j := \mu_j/V$ denote the j 'th component of the center of mass. For $j \neq 3$,

$$c_j = \frac{\lambda}{4} \frac{4h - \lambda(a_j + a_1 + a_2)}{3h - \lambda(a_1 + a_2)}.$$

Since $\gamma_1 + \gamma_2 + \gamma_3 = 0$, a straightforward calculation shows that

$$\langle H, c_Y \rangle = \sum_{i=1}^2 (\gamma_3 - \gamma_i) c_i = \lambda \left(\gamma_3 + \frac{\lambda(\gamma_1 a_1 + \gamma_2 a_2)}{12h - 4\lambda(a_1 + a_2)} \right).$$

This is a linear function of h and λ exactly if $\gamma_1 a_1 + \gamma_2 a_2 = 0$. Hence if H is mass linear, this condition must be satisfied.

To prove the converse, assume that $\gamma_1 a_1 + \gamma_2 a_2 = 0$. Given $\kappa \in \mathcal{C}_a$, note that by Equation (2.6) in Lemma 2.6,

$$Y(\kappa) = Y(0, 0, \lambda, 0, h) - (\kappa_1, \kappa_2, \kappa_4),$$

where $\lambda = \kappa_1 + \kappa_2 + \kappa_3$ and $h = \kappa_4 + \kappa_5 + a_1 \kappa_1 + a_2 \kappa_2$.

Hence,

$$\langle H, c_Y(\kappa) \rangle = \langle H, c_Y(0, 0, \lambda, 0, h) \rangle - \langle H, (\kappa_1, \kappa_2, \kappa_4) \rangle = \sum_{i=1}^3 \kappa_i \gamma_i,$$

as required. □

Remark 4.9. Lemma 4.8 implies the following fact. If $H = \gamma_1 \eta_1 + \gamma_2 \eta_2 + \gamma_3 \eta_3$ is mass linear on Y , where $\gamma_1 + \gamma_2 + \gamma_3 = 0$, then $\langle H, c_Y \rangle = \sum_{i=1}^3 \gamma_i \kappa_i$. In fact, there is an easier proof of this fact. As we saw in Example 3.12, Δ is a Δ_2 bundle over Δ_1 . By Lemma 3.21, the base facets F_4 and F_5 are symmetric. By Proposition 2.9 the restriction of H to F_4 is mass linear and the coefficient of the support number of F_i in $\langle H, c_Y \rangle$ is the coefficient of the support number of $F_i \cap F_4$ in $\langle H, c_{F_4} \rangle$ for all $1 \leq i \leq 3$. But since F_4 is the 2-simplex, every H is inessential. The claim now follows from Proposition 1.18.

We are now ready to prove our main proposition.

Proof of Proposition 4.6. First, note that every $H \in \mathfrak{t}$ can be written uniquely as $H = \sum_{i=1}^5 \gamma_i \eta_i$, where $\gamma_1 + \gamma_2 + \gamma_3 = 0$ and $\gamma_4 + \gamma_5 = 0$. (This holds because $\eta_1 + \eta_2 + \eta_3 = 0$ and $\eta_4 + \eta_5$ is in the span of η_1 and η_2 .) By Lemma 3.7 (or by Example 1.17), F_4 and F_5 are equivalent, and so $\gamma_4 \eta_4 + \gamma_5 \eta_5$ is inessential and

$$\langle \gamma_4 \eta_4 + \gamma_5 \eta_5, c_Y \rangle = \gamma_4 \kappa_4 + \gamma_5 \kappa_5$$

by Proposition 1.18. On the other hand, by Lemma 4.8, $\tilde{H} := \gamma_1 \eta_1 + \gamma_2 \eta_2 + \gamma_3 \eta_3$ is mass linear exactly if $a_1 \gamma_1 + a_2 \gamma_2 = 0$, in which case

$$\langle \tilde{H}, c_Y \rangle = \gamma_1 \kappa_1 + \gamma_2 \kappa_2 + \gamma_3 \kappa_3.$$

The first claim follows immediately.

To prove the last statement, it is convenient to introduce a dummy variable a_3 that we set equal to 0. Then, by Lemma 3.7, for any $\{i, j\} \subset \{1, 2, 3\}$, F_i is equivalent to F_j exactly if $a_i = a_j$. Consider a mass linear function $H = \sum_{i=1}^5 \gamma_i \eta_i$, where $\gamma_1 + \gamma_2 + \gamma_3 = 0$ and $\gamma_4 + \gamma_5 = 0$ as above. Since $\gamma_4 \eta_4 + \gamma_5 \eta_5$ is inessential, H is inessential exactly if $\tilde{H} := \gamma_1 \eta_1 + \gamma_2 \eta_2 + \gamma_3 \eta_3$ is inessential. But if $a_1 = a_2$, for example, then we must have $\gamma_1 + \gamma_2 = 0$ and hence also $\gamma_3 = 0$ so that \tilde{H} is inessential. The cases where $a_1 = a_3$ or $a_2 = a_3$ follow similarly. In contrast, if $a_1 a_2 (a_1 - a_2) \neq 0$ then every mass linear H is essential unless $\gamma_1 = \gamma_2 = \gamma_3$. \square

4.4. Smooth 3-dimensional polytopes. In this subsection we describe mass linear functions on 3-dimensional smooth polytopes. In particular, we show that the only 3-dimensional smooth polytopes which admit essential mass linear functions are Δ_2 bundles over Δ_1 .

We will need the following lemma. Note that the claim is false without the smoothness assumption; this lemma is the one extra piece of information that we use in the smooth case.

Lemma 4.10. *Let Δ be a smooth polytope which is combinatorially equivalent to $\Delta_k \times \Delta_n$. Then Δ is either a Δ_k bundle over Δ_n , or a Δ_n bundle over Δ_k .*

Proof. By assumption, we can label the facets F_1, \dots, F_{k+1} and F'_1, \dots, F'_{n+1} so that $F_I \cap F'_J \neq \emptyset$ for any proper subsets $I \subset \{1, \dots, k+1\}$ and $J \subset \{1, \dots, n+1\}$. Denote the corresponding conormals by $\eta_1, \dots, \eta_{k+1}$ and $\eta'_1, \dots, \eta'_{n+1}$. Given $i \in \{1, \dots, k+1\}$ and $j \in \{1, \dots, n+1\}$, let $I_i = \{1, \dots, k+1\} \setminus \{i\}$ and $J_j = \{1, \dots, n+1\} \setminus \{j\}$. Since $F_{I_{k+1}} \cap F'_{J_{n+1}} \neq \emptyset$, the vectors η_1, \dots, η_k and η'_1, \dots, η'_n form a basis for \mathfrak{t} (and thus identify \mathfrak{t} with \mathbb{R}^{k+n}). Write

$$\eta_{k+1} = \sum_{i=1}^k a_i \eta_i + \sum_{j=1}^n a'_j \eta'_j \quad \text{and} \quad \eta'_{n+1} = \sum_{i=1}^k b_i \eta_i + \sum_{j=1}^n b'_j \eta'_j.$$

Given $i \in \{1, \dots, k+1\}$ and $j \in \{1, \dots, n+1\}$, let $A_{i,j}$ denote the matrix with columns $\eta_1, \dots, \eta_{i-1}, \eta_{i+1}, \dots, \eta_{k+1}, \eta'_1, \dots, \eta'_{j-1}, \eta'_{j+1}, \dots, \eta'_{n+1}$. Since $F_{I_i} \cap F'_{J_j} \neq \emptyset$ and Δ is smooth, $\det A_{i,j} = \pm 1$. In fact, $\det A_{i,j} = (-1)^{(k+1-i)+(n+1-j)}$. In particular, for all $i \in \{1, \dots, k\}$,

$\det A_{i,n+1} = (-1)^{k+1-i}$; therefore, $a_i = -1$ for all such i . A similar argument shows that $b'_j = -1$ for all j .

It remains to show that either $a'_j = 0$ for all $j \in \{1, \dots, n\}$ or $b_i = 0$ for all $i \in \{1, \dots, k\}$. Suppose not. Then there exists $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n\}$ such that $b_i a'_j \neq 0$. On the other hand, since the matrix $A_{i,j}$ has determinant $(-1)^{(k-i)+(n-j)}$,

$$\det \begin{pmatrix} -1 & a'_j \\ b_i & -1 \end{pmatrix} = 1.$$

But then $b_i a'_j = 0$, a contradiction. \square

Lemma 4.11. *Let $H \in \mathfrak{t}$ be a mass linear function on a smooth 3-dimensional polytope $\Delta \subset \mathfrak{t}^*$ with more than two asymmetric facets. If every asymmetric facet is pervasive, then one of the following statements holds:*

- Δ is the simplex Δ_3 .
- Δ is a Δ_1 bundle over Δ_2 ; the base facets are the asymmetric facets.
- Δ is a Δ_2 bundle over Δ_1 ; the fiber facets are the asymmetric facets.

Proof. Let A be the set of asymmetric facets and let S be the set of symmetric facets. By Lemma 4.1, Δ has no symmetric edges. Since the intersection of two symmetric facets is by definition a symmetric face, this implies that the symmetric facets are disjoint. Since every asymmetric facet is pervasive, there are $\frac{1}{2}|A|(|A| - 1) + |A||S|$ edges. Since the polytope is simple and 3-dimensional, there are two-thirds as many vertices. Since the Euler number of Δ is 2,

$$\begin{aligned} |A| + |S| - \frac{1}{6}|A|(|A| - 1) - \frac{1}{3}|A||S| &= 2, \quad \text{that is,} \\ (|A| - 3)((|A| + 2|S| - 4)) &= 0. \end{aligned}$$

Assume first that $|A| + 2|S| = 4$. Since every 3-dimensional polytope has at least four facets, this implies that $|A| = 4$ and $|S| = 0$; hence Δ is the simplex Δ_3 .

So assume that $|A| = 3$; label the asymmetric facets F_1, F_2 , and F_3 . Since no 3-dimensional polytope has only three facets, $|S| \neq 0$. Since every symmetric facet G intersects each asymmetric facet, G is a triangle, and hence also intersects $F_1 \cap F_2$, $F_1 \cap F_3$, and $F_2 \cap F_3$. Since the edge $F_1 \cap F_2$ has two vertices, $|S|$ is 1 or 2. In the first case, Δ is again a tetrahedron. In the second case, Δ is combinatorially equivalent to the product $\Delta_2 \times \Delta_1$. The claim now follows from Lemma 4.10. \square

Remark 4.12. More generally, let Δ be a 3-dimensional simple polytope; assume that every facet is pervasive. As in the argument above, the fact that there are two-thirds as many vertices as edges and that the Euler number of Δ is 2 implies that Δ is combinatorially equivalent to a 3-simplex.

In contrast, there are many 4-dimensional simple polytopes with the property that every facet is pervasive. For example, $C_4(N)$, the 4-dimensional cyclic polytope with N vertices is simplicial (each facet contains exactly 4 vertices) and 2-neighborly (each pair of vertices is joined by an edge); cf. [9]. Therefore, its dual polytope is a 4-dimensional simple polytope with N facets, each of which is pervasive.

We are now ready to classify 3-dimensional smooth polytopes with essential mass linear functions.

Proof of Theorem 1.4. By Lemma 2.13, if $H \neq 0$, then Δ has at least two asymmetric facets. If it has exactly two, then the claim follows by Corollary 4.5. So assume that Δ has more than two asymmetric facets. By Proposition 3.25, after possibly subtracting an inessential function, we may assume that every asymmetric facet is pervasive. Hence, by Lemma 4.11, Δ is either the simplex Δ_3 , a Δ_1 bundle over Δ_2 with symmetric fiber facets, or a Δ_2 bundle over Δ_1 with symmetric base facets. In either of the first two cases, the asymmetric facets are equivalent. Hence H is inessential by Corollary 3.20. Therefore, Δ is a Δ_2 bundle over Δ_1 , that is, Δ is the polytope described in Example 1.1. \square

With a little more effort, we can give a complete list of all smooth 3-dimensional polytopes which admit nonconstant mass linear functions.

Theorem 4.13. *Let $H \in \mathfrak{t}$ be a nonzero mass linear function on a smooth 3-dimensional polytope $\Delta \subset \mathfrak{t}^*$. One of the following statements holds:*

- Δ is a bundle over a simplex.
- Δ is a 1-fold expansion.

Proof. The 3-simplex Δ_3 is the expansion of the 2-simplex Δ_2 along an edge. Moreover, a Δ_2 bundle over Δ_1 , a Δ_1 bundle over $\Delta_1 \times \Delta_1$, and the product $\Delta_1 \times \Delta_1 \times \Delta_1$ are all bundles over the 1-simplex Δ_1 . Therefore, this result follows immediately from Proposition 4.14 below. \square

Proposition 4.14. *Let $H \in \mathfrak{t}$ be a nonzero mass linear function on a smooth 3-dimensional polytope $\Delta \subset \mathfrak{t}^*$. One (or more) of the following statements holds:*

- Δ is a bundle over Δ_1 ; the base facets are the asymmetric facets.
- Δ is a 1-fold expansion; the base-type facets are the asymmetric facets.
- Δ is the simplex Δ_3 .
- Δ is a Δ_1 bundle over Δ_2 ; the base facets are the asymmetric facets.
- Δ is a Δ_2 bundle over Δ_1 ; if either base facet is asymmetric then both are.
- Δ is a Δ_1 bundle over $\Delta_1 \times \Delta_1$; the base facets are the asymmetric facets.
- Δ is the product $\Delta_1 \times \Delta_1 \times \Delta_1$; every facet is asymmetric.

Proof. By Lemma 2.13, Δ has at least two asymmetric facets. If it has exactly two, then the claim follows by Corollary 4.5. So assume that Δ has more than two asymmetric facets.

If all the asymmetric facets are pervasive, then the claim holds by Lemma 4.11. So assume that there is an asymmetric facet F which is not pervasive.

Corollary 3.24 implies that Δ is bundle over Δ_1 with base facets F and G . By Lemma 2.8, G is also asymmetric. By Proposition 3.22, there exists a mass linear function $\tilde{H} \in \mathfrak{t}$ so that $\tilde{H} - H$ is inessential, the base facets are \tilde{H} -symmetric, and at least one fiber facet is still \tilde{H} -asymmetric. By Proposition 2.9, the restriction of \tilde{H} to the 2-dimensional polygon F is mass linear and at least one edge of F is \tilde{H} -asymmetric. By Proposition 4.2, this implies that F has at most four edges.

If F is a triangle, then Δ is a Δ_2 bundle over Δ_1 and both base facets are H -asymmetric (although they are \tilde{H} -symmetric).

If F is a quadrilateral, Δ is combinatorially equivalent to the product $\Delta_1 \times \Delta_1 \times \Delta_1$. Suppose that Δ has an H -symmetric facet. Since every symmetric facet intersects every asymmetric facet and there are at least three asymmetric facets, this implies that Δ has exactly two symmetric facets and they do not intersect. Since none of the facets are pervasive, Proposition 2.11 implies that every asymmetric facet is flat, that is, there exists a hyperplane which contains the conormal of every facet that meets it. If F and F' are asymmetric facets which intersect, then the associated hyperplanes must be distinct, and so they intersect in a line L . Since both symmetric facets intersect F and F' , the two symmetric faces have conormals in L and hence are parallel. Moreover each is affine equivalent to the product $\Delta_1 \times \Delta_1$. Hence, Δ is a Δ_1 bundle over $\Delta_1 \times \Delta_1$ and the fiber facets are symmetric. On the other hand, if none of the facets are H -symmetric then a very similar argument proves that every pair of opposite faces is parallel. Hence, Δ is the product $\Delta_1 \times \Delta_1 \times \Delta_1$. \square

We will need the following result in the next paper.

Lemma 4.15. *Let $H \in \mathfrak{t}$ be a mass linear function on a smooth 3-dimensional polytope $\Delta \subset \mathfrak{t}^*$. Then*

$$\sum_{i=1}^N \gamma_i = 0,$$

where γ_i is the coefficient of the support number of the facet F_i in the linear function $\langle H, c_\Delta \rangle$.

Proof. If H is inessential, then $\sum_i \gamma_i = 0$ by Proposition 1.18. Otherwise, by Theorem 1.4, Δ is a Δ_2 bundle over Δ_1 . The result now follows immediately from Proposition 4.6. (See also Remark 4.9.) \square

5. RELATIONSHIP TO GEOMETRY

We begin by constructing the symplectic toric manifold associated to each smooth polytope. Let $\Delta = \bigcap_{i=1}^N \{x \in \mathfrak{t}^* \mid \langle \eta_i, x \rangle \leq \kappa_i\}$ be a smooth polytope with facets F_1, \dots, F_N . Let e_1, \dots, e_N denote the standard basis for \mathbb{R}^N , and e_1^*, \dots, e_N^* denote the dual basis for $(\mathbb{R}^N)^*$. Let $\pi: \mathbb{R}^N \rightarrow \mathfrak{t}$ denote the linear function given by $\pi(e_i) = -\eta_i$ for all i ; let \mathfrak{k} denote its kernel. Since Δ is bounded, π is surjective. Since $\eta_i \in \ell$ for all i , π induces a surjective map from $(S^1)^N$ to $T := \mathfrak{t}/\ell$; since η_1, \dots, η_N generate ℓ , the kernel K of this map is connected. Thus we have dual short exact sequences of vector spaces and a short exact sequence of groups

$$(5.1) \quad 0 \rightarrow \mathfrak{k} \xrightarrow{\iota} \mathbb{R}^N \xrightarrow{\pi} \mathfrak{t} \rightarrow 0,$$

$$(5.2) \quad 0 \rightarrow \mathfrak{t}^* \xrightarrow{\pi^*} (\mathbb{R}^N)^* \xrightarrow{\iota^*} \mathfrak{t}^* \rightarrow 0, \quad \text{and}$$

$$(5.3) \quad 1 \rightarrow K \xrightarrow{\iota} (S^1)^N \rightarrow T \rightarrow 1.$$

The moment map $\mu: \mathbb{C}^N \rightarrow \mathfrak{k}^*$ for the natural K action on \mathbb{C}^N is given by $\mu(z) = \iota^* \left(\frac{1}{2}|z_1|^2, \dots, \frac{1}{2}|z_N|^2 \right)$. Thus the image $\mu(\mathbb{C}^N)$ is the projection of the positive quadrant in $(\mathbb{R}^N)^*$ and so has outward conormals $\pi(-e_1) = \eta_1, \dots, \pi(-e_N) = \eta_N$. Let $(M_\Delta, \omega_\Delta)$ be the symplectic quotient $\mathbb{C}^N // K := \mu^{-1}(\iota^*(\kappa))/K$. The $(S^1)^N$ action on \mathbb{C}^N descends to an effective T action on M_Δ ; the moment map $\Phi_\Delta: M_\Delta \rightarrow \mathfrak{t}^*$ for this action is given by $\Phi_\Delta([z]) = (\pi^*)^{-1} \left(\frac{1}{2}|z_1|^2 - \kappa_1, \dots, \frac{1}{2}|z_N|^2 - \kappa_N \right)$ for all $[z] \in M_\Delta$. Hence, $\Phi_\Delta(M_\Delta) = \Delta$, as required.

Note also that M_Δ has a canonical ω_Δ -compatible complex structure J_Δ , and hence has a natural Kähler structure. To define J_Δ , denote by (ω_0, J_0) the standard symplectic form and complex structure on \mathbb{C}^N and define $V := \mu^{-1}(\iota^*(\kappa))$. Then the bundle $TV \cap J_0(TV)$ is ω_0 -orthogonal to the K -orbits and its complex structure descends to J_Δ on $T(M_\Delta)$. Moreover, J_Δ is unique up to a T -equivariant Kähler isometry.⁷

Remark 5.1. There is an alternate way to construct the manifold M_Δ described above. Given $z \in \mathbb{C}^N$, let I_z be the set of $i \in \{1, \dots, N\}$ so that $z_i = 0$. Define a set

$$\mathcal{U} := \left\{ z \in \mathbb{C}^N \mid \bigcap_{i \in I_z} F_i \neq \emptyset \right\}.$$

As a complex manifold, $M_\Delta = \mathcal{U}/K_{\mathbb{C}}$, where $K_{\mathbb{C}}$ denotes the complexification of K . Moreover, under this quotient map, the i 'th coordinate hyperplane in \mathbb{C}^N corresponds to $\Phi^{-1}(F_i)$, the preimage of the i 'th facet under the moment map. For more details, see Audin [1, Ch VII] or Cox–Katz [5]. From this description, it is clear that J_Δ does not depend on κ .

The cohomology $H^2(M; \mathbb{R})$ is naturally isomorphic to \mathfrak{k}^* . Under this isomorphism, the cohomology class of the symplectic form ω corresponds to $\iota^*(\kappa) \in \mathfrak{k}^*$. The integral $\frac{1}{n!} \int_M \omega^n$ is (up to a normalizing constant) the volume of Δ with respect to standard Lebesgue measure.

Many of the terms that we introduce for polytopes have geometric interpretations. For example, a facet F of a smooth polytope Δ is flat exactly if the corresponding divisor $\Phi^{-1}(F)$ in M has self-intersection zero.

Remark 5.2. Polytopes which are bundles in the sense of Definition 3.10 correspond to bundles of symplectic toric manifolds. To see this, let $\Delta \subset \mathfrak{t}^*$ be a bundle with fiber $\tilde{\Delta} \subset \tilde{\mathfrak{t}}^*$ over the base $\hat{\Delta} \subset \hat{\mathfrak{t}}^*$; we will use the notation of Definition 3.10. Construct the associated toric manifolds $M_\Delta = \mathcal{U}/K_{\mathbb{C}}$, $M_{\tilde{\Delta}} = \tilde{\mathcal{U}}/\tilde{K}_{\mathbb{C}}$, and $M_{\hat{\Delta}} = \hat{\mathcal{U}}/\hat{K}_{\mathbb{C}}$ as in Remark 5.1; identify $\mathbb{C}^{\tilde{N}+\hat{N}}$ with $\mathbb{C}^{\tilde{N}} \times \mathbb{C}^{\hat{N}}$. Since Δ is combinatorially equivalent to $\tilde{\Delta} \times \hat{\Delta}$, $\mathcal{U} = \tilde{\mathcal{U}} \times \hat{\mathcal{U}}$. Moreover, since $\tilde{\eta}_j' = \iota(\tilde{\eta}_j)$ for all j and $\pi(\hat{\eta}_i') = \hat{\eta}_i$ for all i , \tilde{K} is the intersection of K with $(S^1)^{\tilde{N}} \subset (S^1)^{\tilde{N}} \times (S^1)^{\hat{N}}$, and \hat{K} is the image of K under the natural projection $(S^1)^{\tilde{N}} \times (S^1)^{\hat{N}} \rightarrow (S^1)^{\hat{N}}$; in particular, $\hat{K} = K/\tilde{K}$. Therefore, the natural projection from

⁷Throughout we have assumed that our presentation of Δ is minimal, i.e. that each facet F_i is nonempty. If one inserted “ghost” facets, then the above procedure might give a different Kähler structure. For further details, see for example [2].

$K \subset (S^1)^{\tilde{N}} \times (S^1)^{\hat{N}}$ to $(S^1)^{\tilde{N}}$ induces a homomorphism ρ from $\hat{K} \simeq K/\tilde{K}$ to $\tilde{T} = (S^1)^{\tilde{N}}/\tilde{K}$. The toric manifold M_Δ is the associated bundle

$$M_\Delta = M_{\tilde{\Delta}} \times_{\hat{K}_{\mathbb{C}}} \hat{\mathcal{U}},$$

where ρ describes the action of $\hat{K}_{\mathbb{C}}$ on $M_{\tilde{\Delta}}$.

Example 5.3. Given $a = (a_1, a_2) \in \mathbb{R}^2$, let Y_a be the polytope defined in Example 1.1. As we saw in Example 3.12, Y_a is a Δ_2 bundle over Δ_1 . Recall that the outward conormals η_1, \dots, η_5 are given by

$$(-1, 0, 0), \quad (0, -1, 0), \quad (1, 1, 0), \quad (0, 0, -1), \quad \text{and} \quad (a_1, a_2, 1);$$

these satisfy

$$\eta_1 + \eta_2 + \eta_3 = 0 \quad \text{and} \quad a_1\eta_1 + a_2\eta_2 + \eta_4 + \eta_5 = 0.$$

Thus, by definition of the map π in Equation (5.1), K is the subtorus of $(S^1)^5$ generated by the elements $(1, 1, 1, 0, 0)$ and $(a_1, a_2, 0, 1, 1)$ in \mathbb{Z}^5 . Moreover, by Definition 5.1,

$$\mathcal{U} = (\mathbb{C}^3 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\}).$$

Therefore M_{Y_a} is the $\mathbb{C}P^2$ bundle

$$\mathbb{C}P^2 \times_{\mathbb{C}^*} (\mathbb{C}^2 \setminus \{0\})$$

over $\mathbb{C}P^1$, where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ acts on $\mathbb{C}P^2$ by

$$\lambda \cdot [z_1 : z_2 : z_3] = [\lambda^{a_1} z_1 : \lambda^{a_2} z_2 : z_3],$$

that is, $M_{Y_a} = \mathbb{P}(\mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \mathcal{O}(0))$.

Remark 5.4. Polytopes which are 1-fold expansions in the sense of Definition 3.13 correspond to symplectic toric manifolds which are smooth symplectic pencils; that is, they are swept out by a family $\tilde{M}_\lambda, \lambda \in \mathbb{C}P^1$, of (real) codimension 2 symplectic submanifolds (often called fibers) that intersect transversely along a codimension 4 symplectic submanifold called the axis or base locus. To see this, let $\Delta \subset \mathfrak{t}^*$ be the 1-fold expansion of a polytope $\tilde{\Delta} \subset \tilde{\mathfrak{t}}^*$. Construct the associated toric manifolds $M_\Delta = \mathcal{U}/K_{\mathbb{C}}$ and $M_{\tilde{\Delta}} = \tilde{\mathcal{U}}/\tilde{K}_{\mathbb{C}}$, as in Remark 5.1. We will use the notation of Definition 3.13 and identify \mathbb{C}^N with $\mathbb{C}^2 \times \mathbb{C}^{N-2}$. There is a well-defined map from $\{z \in \mathcal{U} \mid z_1 z_2 \neq 0\}$ to $\mathbb{C}^2 \setminus \{0\}$ given by $z \mapsto (z_1, z_2)$. Since the $\tilde{\eta}_j$ all lie in the hyperplane $\tilde{\mathfrak{t}}$, this descends to an equivariant holomorphic map from $M_\Delta \setminus \Phi^{-1}(\hat{F}_1 \cap \hat{F}_2)$ to $\mathbb{C}P^1$. Therefore, M_Δ is a pencil with axis $\Phi^{-1}(\hat{F}_1 \cap \hat{F}_2)$. Notice that the toric manifold obtained by blowing up M_Δ along this axis as in Remark 3.15 is a fibration over $\mathbb{C}P^1$ with fiber $M_{\tilde{\Delta}}$.

Our proof of Proposition 1.22 is based on Weinstein's action homomorphism

$$\mathcal{A}_\omega : \pi_1(\text{Ham}(M, \omega)) \rightarrow \mathbb{R}/\mathcal{P}_\omega.$$

Here, $\text{Ham}(M, \omega)$ denotes the group of Hamiltonian symplectomorphisms of (M, ω) and $\mathcal{P}_\omega \subset \mathbb{R}$, the period group of ω , is the image of $\pi_2(M)$ under the homomorphism $\alpha \mapsto \int_\alpha \omega$.

The homomorphism \mathcal{A}_ω is defined as follows: If the loop $\{\phi_t\}_{t \in S^1}$ in $\text{Ham}(M, \omega)$ is generated by the mean normalized Hamiltonian⁸ K_t for $t \in [0, 1]$, then

$$\mathcal{A}_\omega(\{\phi_t\}) := \int_0^1 K_t(\phi_t(p)) dt - \int_D u^*(\omega)$$

where p is any point in M and $u: D^2 \rightarrow M$ is any smooth map such that $u(e^{2\pi it}) = \phi_t(p)$. If the loop $\{\phi_t\}_{t \in S^1}$ is a circle subgroup $\Lambda \subset \text{Ham}(M, \omega)$ generated by a mean normalized Hamiltonian $K: M \rightarrow \mathbb{R}$ then we may take p to be a fixed point of the action and D to be the constant disc. Hence $\mathcal{A}_\omega(\Lambda)$ is the image in $\mathbb{R}/\mathcal{P}_\omega$ of any critical value of K . This is well defined because for any fixed points p and p' the difference $K(p) - K(p')$ is the integral of ω over the 2-sphere formed by rotating an arc from p to p' by the S^1 -action.

Proof of Proposition 1.22. First, note that since toric manifolds are simply connected, $\text{Symp}_0(M, \omega)$ is $\text{Ham}(M, \omega)$. Given $H \in \ell$, the mean normalized Hamiltonian for the action of Λ_H on M is

$$K = \langle H, \Phi \rangle - \langle H, c_\Delta \rangle,$$

where c_Δ denotes the center of mass of Δ . Since vertices of Δ correspond to fixed points in M , this implies that

$$\mathcal{A}_\omega(\Lambda_H) = \langle H, v \rangle - \langle H, c_\Delta \rangle \in \mathbb{R}/\mathcal{P}_\omega$$

for any vertex v of Δ .

Clearly if Λ_H vanishes in $\pi_1(\text{Symp}_0(M, \omega))$, then $\mathcal{A}_\omega(\Lambda_H) = 0$. In fact, it is immediate from Moser's homotopy argument that if Λ_H contracts in $\text{Symp}_0(M, \omega)$, then it also contracts for all sufficiently small perturbations of ω . Since changing κ corresponds to changing the symplectic form on M , for any vertex v of Δ the image of Λ_H under the action homomorphism

$$\mathcal{A}_{\omega(\kappa)}(\Lambda_H) = \langle H, v \rangle - \langle H, c_\Delta(\kappa) \rangle$$

must lie in $\mathcal{P}_{\omega(\kappa)}$ for all κ in some open set of \mathbb{R}^N . Since $\mathcal{P}_{\omega(\kappa)}$ is generated by the lengths of the edges of the polytope Δ (considered as a multiple of a primitive vector), it is a finitely generated subgroup of \mathbb{R} whose generators are linear functions of the κ_i with integer coefficients. Similarly, $\langle H, v \rangle$ is a linear function of the κ_i with integer coefficients. Since the function $\kappa \mapsto \mathcal{A}_{\omega(\kappa)}(\Lambda_H)$ is continuous, it follows that the function $\kappa \mapsto \langle H, c_\Delta(\kappa) \rangle$ is also a linear function of the κ_i with integer coefficients as κ varies in some open set. By Lemma 2.3, this implies that it is a linear function of κ_i with integer coefficients for all $\kappa \in \mathcal{C}_K$. \square

The next proposition gives a geometric interpretation of the equivalence relation on the facets of Δ in terms of $\text{Isom}_0(M)$, the identity component of the group of isometries of (M, g_J) . Recall that these act by biholomorphisms of (M, J) and hence also preserve ω . To see this, note that the Kähler condition implies that J is invariant under parallel translation with respect to the Levi-Civita connection of g_J , while, for each path $\phi_t, t \in [0, 1]$, in

⁸ We define the sign of K_t by requiring that $\dot{\phi}_t = X_{K_t}$ where $\omega(X_{K_t}, \cdot) = -dK_t$. We say that K is **mean normalized** iff $\int_M K \omega^n = 0$.

$\text{Isom}_0(M)$ with $\phi_0 = \text{id}$ and each $x \in M$, the derivative $d\phi_t(x) : T_x M \rightarrow T_{\phi_t(x)} M$ is the linear map given by parallel translation along the path $\phi_s(x)$, $s \in [0, t]$.

Proposition 5.5. *Let (M, ω, T, Φ) be an n -dimensional symplectic toric manifold with moment polytope $\Delta \subset \mathfrak{t}^*$, and let $\text{Isom}_0(M)$ be the identity component of the associated Kähler isometry group. Let \mathcal{I} denote the set of equivalence classes of facets of Δ , and let K be defined as in (5.3). Then*

$$\text{Isom}_0(M) \cong \left(\prod_{I \in \mathcal{I}} U(|I|) \right) / K,$$

where for each equivalence class $I \in \mathcal{I}$, the unitary group $U(|I|) \subset U(N)$ acts on the subspace of \mathbb{C}^N spanned by $\{e_i\}_{i \in I}$. Under this identification, $T = (S^1)^N / K$ and $\prod_{I \in \mathcal{I}} U(|I|)$ is the centralizer of K in $U(N)$. Moreover, $\text{Isom}_0(M)$ is a maximal connected compact subgroup of $\text{Symp}_0(M, \omega)$.

Proof. Let $(M_\Delta, \omega_\Delta, J_\Delta, \Phi_\Delta) = \mathbb{C}^N // K$ be the Kähler toric manifold associated to the moment polytope $\Delta \subset \mathfrak{t}^*$, as above. Since M and M_Δ are equivariantly symplectomorphic by [6], we may identify M and M_Δ .

Let $Z(K)$ denote the centralizer of $K \subset (S^1)^N$ in $U(N)$. Our first claim is that $Z(K)$ is the product $\prod_{I \in \mathcal{I}} U(|I|)$. To see this, note first that since $\eta_k = \pi(e_k)$ for all k , $\xi \in \mathfrak{t}^*$ satisfies

$$\langle \eta_i, \xi \rangle = 1 = -\langle \eta_j, \xi \rangle \quad \text{and otherwise} \quad \langle \eta_k, \xi \rangle = 0 \quad \forall k$$

exactly if

$$\langle e_i, \pi^*(\xi) \rangle = 1 = -\langle e_j, \pi^*(\xi) \rangle \quad \text{and otherwise} \quad \langle e_k, \pi^*(\xi) \rangle = 0 \quad \forall k,$$

that is, exactly if $\pi(\xi) = e_i^* - e_j^*$. Therefore, Lemma 3.7 implies that two facets F_i and F_j of Δ are equivalent exactly if $e_i^* - e_j^* \in \pi^*(\mathfrak{t}^*)$. But this holds exactly if $\iota^*(e_i^* - e_j^*) = 0$, that is, iff $\iota^*(e_i^*) = \iota^*(e_j^*)$. Since the weights for the K action on \mathbb{C}^N are exactly $\iota^*(e_1^*), \dots, \iota^*(e_N^*)$, the claim follows immediately.

Now observe that $Z(K) \subset U(N)$ acts on \mathbb{C}^N , preserving the standard symplectic and complex structures on \mathbb{C}^N . This implies that the quotient group $Z(K)/K$ acts on M_Δ , preserving ω_Δ and J_Δ and hence also the Kähler metric g_J . Thus, because $Z(K)$ is connected, $Z(K)/K \subset \text{Isom}_0(M_\Delta)$. But $\text{Isom}_0(M_\Delta)$ is a compact connected Lie group. Therefore, to complete the proof it is enough to prove that $Z(K)/K$ is a maximal compact connected subgroup of $\text{Symp}_0(M_\Delta, \omega_\Delta)$.

To this end, let G be any connected compact subgroup of $\text{Symp}_0(M_\Delta, \omega_\Delta)$ that contains $Z(K)/K$. Then G is a Lie group. By Lemma 5.6 below, T is the maximal torus of G and the Weyl group of G acts on \mathfrak{t}^* as a subgroup of the group of robust symmetries of Δ . By the same argument, T is the maximal torus of $Z(K)/K$ and the Weyl group of $Z(K)/K$ acts as a subgroup of the group of robust symmetries of Δ . Moreover, given any two equivalent facets F_i and F_j , it is easy to check that there is an element of the Weyl group of $Z(K)/K$ which exchanges F_i and F_j and otherwise takes each facet to itself. Therefore, the Weyl group of $Z(K)/K$ acts on \mathfrak{t}^* as the group of robust symmetries of Δ . Since, $Z(K)/K \subset G$, this implies that the two groups have the same Weyl group. But,

because $Z(K)/K = \prod_{I \in \mathcal{I}} U(|I|)/K$, each of its simple factors has a root system of type $A_{|I|-1}$. Therefore, there is no way to extend its root system without either increasing the rank or enlarging the Weyl group. It follows that G has the same roots as $Z(K)/K$, so that the two groups must coincide. \square

Lemma 5.6. *Let (M, ω, T, Φ) be a symplectic toric manifold. Let $G \subset \text{Symp}(M, \omega)$ be a compact connected Lie subgroup that contains T . Then T is a maximal torus of G , and the Weyl group W of G acts on \mathfrak{t}^* as a subgroup of the group of robust symmetries of the moment polytope $\Delta \subset \mathfrak{t}^*$.*

Proof. Since M is simply connected, there is a moment map $\Phi_G: M \rightarrow \mathfrak{g}^*$ for the G action. Since Φ_G is equivariant, the moment image $\Phi_G(M) \subset \mathfrak{g}^*$ is invariant under the coadjoint action of G on \mathfrak{g}^* .

Let $\iota: \mathfrak{t} \rightarrow \mathfrak{g}$ denote the inclusion map. The dual map $\iota^*: \mathfrak{g}^* \rightarrow \mathfrak{t}^*$ is equivariant with respect to the coadjoint action of $N(T)$, where $N(T)$ denotes the normalizer of T in G . Moreover, by adding a constant if necessary, we may assume that $\Phi_T = \iota^* \circ \Phi_G$. Therefore, the moment polytope $\Delta = \Phi_T(M) \subset \mathfrak{t}^*$ is also invariant under the coadjoint action of $N(T)$; hence, $\Phi_T(M)$ is invariant under the action of the Weyl group $W = N(T)/T$.

Finally, write the moment polytope as $\Delta = \bigcap_{i=1}^N \{x \in \mathfrak{t}^* \mid \langle \eta_i, x \rangle \leq \kappa_i\}$, where N is the number of its (nonempty) facets. By the construction explained in the beginning of this section, for all $\kappa' \in \mathcal{C}_\Delta$, there exists a T -invariant symplectic form $\omega_{\kappa'}$ with corresponding moment polytope $\Delta(\kappa')$. Let $\alpha \in \Omega^2(M; \mathbb{R})$ be any G -invariant 2-form in the cohomology class $[\omega' - \omega]$. For $\epsilon > 0$ sufficiently small, the form $\omega + \epsilon\alpha$ is a G -invariant symplectic form with moment polytope $\Delta((1 - \epsilon)\kappa + \epsilon\kappa')$. By the previous paragraph, this implies that, up to translation, this polytope is also invariant under the action of the Weyl group. The conclusion now follows from Corollary 3.5. \square

Proof of Lemma 1.27. Let $Z(K) = \prod_{I \in \mathcal{I}} U(|I|)$. Then the natural map from $\pi_1((S^1)^N)$ to $\pi_1(Z(K))$ is surjective. Moreover, $H' = \sum_i \beta_i e_i \in \mathbb{Z}^N$ generates a trivial element of $\pi_1(Z(K)) = \pi_1(\prod_{I \in \mathcal{I}} U(|I|))$ exactly if it lies in the Lie algebra of $\prod_{I \in \mathcal{I}} SU(|I|)$, that is, exactly if $\sum_{i \in I} \beta_i = 0$ for all $i \in I$.

Since K is connected, the natural maps from $\pi_1((S^1)^N)$ to $\pi_1(T)$ and from $\pi_1(Z(K))$ to $\pi_1(Z(K)/K)$ are surjective. Therefore, the natural map from $\pi_1(T)$ to $\pi_1(Z(K)/K)$ is surjective. Moreover, $H \in \ell$ generates a trivial element of $\pi_1(Z(K)/K)$ exactly if there exists $H' = \sum_i \beta_i e_i \in \pi^{-1}(H) \subset \mathbb{Z}^N$ that generates a trivial element of $\pi_1(Z(K))$, where $\pi: \mathbb{R}^N \rightarrow \mathfrak{t}$ is as defined in equation (5.1). But this happens exactly if $H = \sum_i \beta_i \eta_i \in \ell$, where $\beta_i \in \mathbb{Z}$ for all i and $\sum_{i \in I} \beta_i = 0$ for all $I \in \mathcal{I}$. \square

Example 5.7. Given $a = (a_1, a_2) \in \mathbb{R}^2$, let Y_a be the polytope defined in Example 1.1. Construct $M_{Y_a} = \mathcal{U}/K$ as in Example 5.3; let $Z(K)$ be the centralizer of K in $U(N)$. Recall that K is the subtorus of $(S^1)^5$ generated by the elements $(1, 1, 1, 0, 0)$ and $(a_1, a_2, 0, 1, 1)$. Therefore,

- If $a_1 a_2 (a_1 - a_2) \neq 0$, then $Z(K) = S^1 \times S^1 \times S^1 \times U(2)$, and so $\pi_1(Z(K)/K)$ has rank 2.

- If $a_1 = a_2 \neq 0$ (or if $a_1 = 0 \neq a_2$ or $a_1 \neq 0 = a_2$) then $Z(K) = U(2) \times S^1 \times U(2)$ and $\pi_1(Z(K)/K)$ has rank 1.
- If $a_1 = a_2 = 0$ then $Z(K) = U(3) \times U(2)$ and $\pi_1(Z(K)/K)$ has rank 0.

Thus the structure of $Z(K)/K$ depends on the coefficients (a_1, a_2) . This is consistent with the calculation of the equivalence relation on the facets via Lemma 3.7 given in the proof of Proposition 4.6. Also, by Proposition 5.5, $\text{Isom}_0(M) = Z(K)/K$. It is hard to check this independently except in the case $a_1 = a_2 = 0$ when Δ is a product. However, one can verify the consistency of Lemma 1.27, which implies that the set of inessential functions should have a corresponding dependence on (a_1, a_2) . This is the case, as the reader may check by looking at Remark 4.7.

Remark 5.8. (i) A rational polytope Δ defines an algebraic variety $X := X_\Delta$. In this case, it is customary to consider a refinement \sim' of the equivalence relation \sim that takes multiplicities into account. One way of doing this is to consider the Chow group $A_{n-1}(X)$ of divisors in X , and to set $F_i \sim' F_j$ exactly if F_i and F_j have the same image in $A_{n-1}(X)$; cf. Cox [4, §4]. If Δ is smooth then $X = M$, $A_{n-1}(X)$ is naturally identified with $H_{2n-2}(M; \mathbb{Z})$ and $F_i \sim' F_j$ exactly if the submanifolds $\Phi^{-1}(F_i)$ and $\Phi^{-1}(F_j)$ of M are homologous. It then follows from standard facts about toric manifolds that the two equivalence relations are the same. But they differ in general, for example for the polygon with conormals $(-1, 0), (0, -\frac{2}{3}), (1, \frac{2}{3})$ corresponding to the weighted projective space $\mathbb{CP}^2(1, 2, 3)$; cf. Example 3.2.

(ii) Consider the group $\text{Aut}(M)$ of biholomorphisms of the complex manifold (M, J_Δ) . Its structure was worked out by Demazure [7]. As explained in Cox [4], for any rational polytope Δ , its identity component $\text{Aut}_0(X)$ is isomorphic to a semidirect product $R_u \rtimes G_s$ where the unipotent radical R_u is isomorphic to \mathbb{C}^k for some $k \geq 0$ and the reductive part G_s is isomorphic to the product $(\prod_{I \in \mathcal{I}'} GL(|I|, \mathbb{C})) / K_{\mathbb{C}}$, where $K_{\mathbb{C}}$ is the complexification of K and \mathcal{I}' is the set of equivalence classes with respect to the relation \sim' discussed in (i). Moreover, the elements of $\pi_0(\text{Aut}(X))$ act nontrivially on homology.

Let us call a facet F_i of Δ **convex** if all the conormals except for η_i lie in a closed half-space of \mathfrak{t} . It follows easily from the description of Demazure roots in [4, §4] that in the smooth case the unipotent radical R_u is trivial (i.e. $\text{Aut}(M)$ is reductive) exactly if each convex facet of Δ is equivalent to at least one other facet. This fact about the structure of $\text{Aut}(M)$ does not seem relevant in our situation, although it is in other geometric contexts; cf. Remark 1.34.

Remark 5.9. The first claim in Proposition 5.5 is that $\text{Isom}_0(M) \cong Z(K)/K$. One can also prove this by thinking of $\text{Isom}_0(M)$ as the identity component of $\text{Symp}_0(M) \cap \text{Aut}_0(M)$, where $\text{Aut}_0(M)$ is as in the previous remark. Since R_u contains no nontrivial compact subgroup, $\text{Isom}_0(M)$ consists of the unitary elements in $(\prod_{I \in \mathcal{I}} GL(|I|, \mathbb{C})) / K_{\mathbb{C}}$ and so is $Z(K)/K$.

APPENDIX A. POWERFUL FACETS

APPENDIX WRITTEN WITH VLADLEN TIMORIN

The purpose of this appendix is to analyze mass linear function on 4-dimensional simple polytopes where every facet is asymmetric. In part II, we will need these results in order to classify mass linear functions on 4-dimensional smooth polytopes. We need one last important definition.

Definition A.1. *We say that a facet F of a simple polytope Δ is **powerful** if it is connected to each vertex of Δ by an edge.*

We will adopt the notations of §2.2. In particular, given $H \in \mathfrak{t}$ and a simple polytope $\Delta \subset \mathfrak{t}^*$, fix an identification of \mathfrak{t}^* with Euclidean space and define

$$V = \int_{\Delta} dx, \quad \mu = \int_{\Delta} H(x) dx, \quad \text{and} \quad \widehat{H}(\kappa) = \langle H, c_{\Delta}(\kappa) \rangle.$$

Let ∂_i denote the operator of differentiation with respect to κ_i .

Proposition A.2. *Let $H \in \mathfrak{t}$ be a mass linear function on a simple polytope $\Delta \subset \mathfrak{t}^*$. Every asymmetric facet is powerful.*

Proof. Let F_1 be an asymmetric facet and let $v = F_J$ be a vertex which is not connected to F_1 by any edge; number the facets so that $J = \{2, \dots, n+1\}$. Since \widehat{H} is linear, its second derivatives all vanish. By assumption, the intersection $F_1 \cap e_j$ is empty for each $j \in J$, where e_j is the edge $\bigcap_{i \in J \setminus \{j\}} F_i$. Hence, by Theorem 2.2, if we apply the differential operator $\partial_1 \cdots \partial_{n+1}$ to the formula $\mu = \widehat{H}V$ we obtain

$$0 = K_v \partial_1 \widehat{H},$$

where K_v is a positive real number. Since F_1 is an asymmetric facet $\partial_1 \widehat{H} \neq 0$; this gives a contradiction. \square

With this motivation, we analyze polytopes with powerful facets.

Lemma A.3. *Let Δ be a simple polytope. If two powerful facets F and G do not intersect, then Δ is combinatorially equivalent to the product $\Delta_1 \times F$. If every facet is powerful, then Δ is combinatorially equivalent to a product $\Delta_1 \times \cdots \times \Delta_1 \times \Delta'$, where every facet of Δ' is both powerful and pervasive.*

Proof. Assume that two powerful facets F and G do not intersect. Then every vertex in F has a unique edge which does not lie in F , and that edge must meet G . Conversely every edge which meets (but does not lie in) G must meet F . Since Δ is connected, this implies that Δ is combinatorially equivalent to $\Delta_1 \times F$.

The second claim follows by induction. \square

Remark A.4. We can now give an alternate proof of Corollary 3.24. Let $H \in \mathfrak{t}$ be a mass linear function on a simple polytope $\Delta \subset \mathfrak{t}^*$. Let F be an asymmetric facet, and let G be any facet which is disjoint from F . By Lemma 2.8, G is also asymmetric. Hence Proposition A.2 implies that F and G are powerful. Next, Lemma A.3 implies that Δ is combinatorially equivalent to $\Delta_1 \times F$.

In the proof of our main proposition we need the following well known technical lemma. We include a proof for the convenience of the reader.

Lemma A.5. *Fix vectors η_1, \dots, η_N in a n -dimensional vector space \mathfrak{t} . Assume that their positive span contains \mathfrak{t} and that no subset of n vectors is linearly dependent. After possibly renumbering, the positive span of $\eta_1, \dots, \eta_{n+1}$ contains \mathfrak{t} .*

Proof. After possibly renumbering, there exists k so that all of \mathfrak{t} lies in the positive span of η_1, \dots, η_k , but not in the positive span of any proper subset. Assume that $k > n + 1$. Write $0 = \sum_{i=1}^k a_i \eta_i$, where $a_i > 0$ for all i . There exists a linear relation $\sum_{i=1}^{n+1} d_i \eta_i = 0$; let $d_i = 0$ for all $n + 1 < i \leq k$. Since $k > n + 1$, by adding an appropriate multiple of (d_1, \dots, d_k) to (a_1, \dots, a_k) , we can find non-negative numbers b_1, \dots, b_k – at least one of which is zero and at least one of which is not – so that $\sum_{i=1}^k b_i \eta_i = 0$. After renumbering again, this implies that $\sum_{i=1}^\ell b_i \eta_i = 0$, where $\ell < k$ and $b_i > 0$ for all i . Since no set of n vectors is linearly dependent, this implies that every $\alpha \in \mathfrak{t}$ can be written as a (not necessarily positive) linear combination $\alpha = \sum_{i=1}^\ell c_i \eta_i$. Hence, for large enough t , α can be written as a positive sum $\alpha = \sum_{i=1}^\ell (c_i + t b_i)$. This contradicts the claim that k is minimal. \square

Proposition A.6. *Let Δ be a n -dimensional simple polytope; assume that every facet is powerful. If $n \leq 4$, then Δ is combinatorially equivalent to the product of simplices.*

Proof. The polytope Δ has at most $2n$ facets. To see this, pick a vertex v of Δ . Since Δ is simple, v lies on n facets and n edges. Since every facet of Δ is powerful, there is an edge from v to every facet that does not contain v .

By Lemma A.3, we may assume that every facet of Δ is pervasive. By Remark 4.12, if $n = 3$ this implies that Δ is combinatorially equivalent to a simplex; the same claim is obvious if $n < 3$ or if $n = 4$ and Δ has five facets. So assume that Δ is 4-dimensional and has N facets, where $6 \leq N \leq 8$.

Write $\Delta = \bigcap_{i=1}^N \{x \in \mathfrak{t}^* \mid \langle \eta_i, x \rangle \leq \kappa_i\}$. Since our statement is purely combinatorial we may slightly perturb the facets so that the polytope is generic. In particular, no four outward conormals are linearly dependent; hence, after possibly renumbering, the lemma above implies that the positive span of η_1, \dots, η_5 contains all of \mathfrak{t} . Now define

$$P := \bigcap_{i=1}^6 \{x \in \mathfrak{t}^* \mid \langle \eta_i, x \rangle \leq \kappa_i\} \quad \text{and} \quad H_\ell^+ := \{x \in \mathfrak{t}^* \mid \langle \eta_\ell, x \rangle \leq \kappa_\ell\} \quad \text{for } 7 \leq \ell \leq N.$$

Clearly,

$$\Delta = P \cap \bigcap_{\ell=7}^N H_\ell^+.$$

By construction, P is bounded; by the genericity assumption, P is simple, as is $P \cap H_\ell^+$ for any ℓ . Since every facet of Δ is pervasive, and every facet of P contains a facet of Δ , the facets of P are also pervasive. Moreover, it is well known that any n -dimensional polytope with $n + 2$ facets, such as P , is combinatorially equivalent to a product of simplices; for a proof in the current setting see [16, Prop 1.1.1]. Thus P is combinatorially equivalent

either to $\Delta_2 \times \Delta_2$ or to $\Delta_1 \times \Delta_3$. Finally, some of the facets of the latter polytope are nonpervasive, and so P is combinatorially equivalent to $\Delta_2 \times \Delta_2$.

This solves the case $N = 6$, so assume that $N > 6$. Label the facets of P as F_k and F'_k for $0 \leq k \leq 2$, so that $F_0 \cap F_1 \cap F_2 = F'_0 \cap F'_1 \cap F'_2 = \emptyset$. Let V be the set of all vertices of P ; label them

$$v_{ij} = \bigcap_{m \neq i} F_m \cap \bigcap_{n \neq j} F'_n.$$

We will think of the vertices as lying on a 3×3 grid, where the first subscript determines the row and the second determines the column.

Now, fix ℓ such that $7 \leq \ell \leq N$. Let $V_\ell^+ = V \cap H_\ell^+$, and let $V_\ell^- \subset V$ be the complement of V_ℓ^+ . Since the facets of Δ are pervasive, the facets $F_0 \cap \Delta$ and $F_1 \cap \Delta$ must intersect. A fortiori, $F_0 \cap F_1 \cap H_\ell^+ \neq \emptyset$. Since H_ℓ^+ is a half space, this implies that (at least) one of the three vertices v_{20}, v_{21} or v_{22} of $F_0 \cap F_1$ lies in V_ℓ^+ . A similar argument implies that V_ℓ^+ contains at least one vertex in each row and at least one in each column.

Moreover, suppose that v_{00} and v_{11} are in V_ℓ^+ . Consider the 2-dimensional face $F_2 \cap F'_2$; it is a quadrilateral with vertices v_{00}, v_{01}, v_{11} , and v_{10} . The intersection of this face with the half space H_ℓ^+ cannot contain just the two opposite vertices v_{00} and v_{11} ; it must also contain (at least) one of the vertices v_{10} or v_{01} . More generally, if V_ℓ^+ contains the vertices v_{ij} and v_{mn} where $i \neq m$ and $j \neq n$, then it also contains v_{in} or v_{mj} . A brief analysis shows that this, together with the paragraph above, implies that V_ℓ^+ contains at least five vertices.

On the other hand, the facets $F_i \cap \Delta$ and $F'_i \cap \Delta$ must also intersect the new facet F_ℓ for each i . Therefore, the facets F_i and F'_i cannot lie entirely in V_ℓ^+ . Hence at least one vertex of F_i and at least one vertex of F'_i lies in V_ℓ^- for each i . It is straightforward to check that this is only possible if V_ℓ^- contains two vertices v_{ij} and v_{mn} , where $i \neq m$ and $j \neq n$. As in the previous paragraph, this implies that V_ℓ^- also contains either v_{in} or v_{mj} . In either case, V_ℓ^- contains two vertices which lie in the same column, and two vertices which lie in the same row.

Now suppose that v_{00} is a vertex of Δ . Every edge of Δ which meets v_{00} is of the form $e \cap \Delta$, where e is an edge of P which meets v_{00} , that is, where e is $\overline{v_{00}v_{10}}$, $\overline{v_{00}v_{20}}$, $\overline{v_{00}v_{01}}$, or $\overline{v_{00}v_{02}}$. Moreover, consider – for example – the edge $e = \overline{v_{00}v_{10}}$. If $e \cap \Delta$ connects v_{00} to $F_0 \cap \Delta$, then v_{10} must lie in Δ ; otherwise $e \cap \Delta$ will not intersect $F_0 \cap \Delta$. On the other hand, if $e \cap \Delta$ connects v_{00} to the new facet F_ℓ of Δ , then v_{10} must lie in V_ℓ^- . Since $F_0 \cap \Delta$ is powerful, there must be an edge from v_{00} to $F_0 \cap \Delta$. This edge must be of the form $e \cap \Delta$, where e is an edge in P from v_{00} to F_0 , that is, $\overline{v_{00}v_{10}}$ or $\overline{v_{00}v_{20}}$. Hence, (at least) one of the vertices v_{10} and v_{20} lies in Δ . On the other hand, the new facet F_ℓ is also powerful. Hence, V_ℓ^- contains (at least) one of the vertices v_{10}, v_{20}, v_{02} , or v_{01} . A similar argument shows that Δ cannot contain exactly one vertex in any row or column, and that if Δ contains v_{ij} then V_ℓ^- contains some vertex which either lies in the same row or in the same column.

We now specialize, and assume that $N = 7$. In this case, v_{ij} lies in Δ exactly if $v_{ij} \in V_7^+$. Therefore, since V_7^+ contains at least one vertex in each column, the paragraph above

implies that it contains at least two vertices in each column. But this contradicts the fact that V_7^- contains two vertices in the same column.

So we may assume that $N = 8$. In this case, v_{ij} lies in Δ exactly if $v_{ij} \in V_7^+ \cap V_8^+$. Hence, $V_7^+ \cap V_8^+$ cannot contain exactly one vertex in any row or column. On the other hand, V_7^- contains two vertices in the same column and also contains two vertices in the same row, so $V_7^+ \cap V_8^+$ cannot contain two or more vertices in every column or in every row. Together, these imply that $V_7^+ \cap V_8^+$ must not contain any vertices in at least one row and at least one column. Assume that these are the last row and column. Since V_7^+ and V_8^+ each contain at least five vertices, $V_7^+ \cap V_8^+$ is not empty. This implies that, $V_7^+ \cap V_8^+ = \{v_{00}, v_{01}, v_{10}, v_{11}\}$. But then V_7^- must contain at least one vertex in the same row or column as each of v_{00}, v_{01}, v_{10} and v_{11} . After possibly swapping the columns with the rows, this implies that V_7^- contains v_{02} and v_{12} . Since V_7^+ contains at least one vertex in every column, this implies that V_7^+ contains v_{22} . A similar argument implies that V_8^+ contains v_{22} . This contradicts the fact that v_{22} does not lie in $V_7^+ \cap V_8^+$. \square

Remark A.7. It is natural to wonder if there is an analog of this result in higher dimensions. Chen [3] recently found a 7-dimensional polytope with all facets pervasive and powerful that is not combinatorially equivalent to a product. However, since his example is not smooth, the following question is still open:

Let Δ be a smooth polytope; assume that every facet is powerful. Is Δ combinatorially equivalent to the product of simplices? If not, is there a structure theorem for such polytopes?

Note that, by Lemma A.3, it is enough to consider the case that every facet of Δ is pervasive. Moreover, the proof above shows that if Δ is n -dimensional, then it has at most $2n$ facets.

Combined, Propositions A.2 and A.6 have the following important corollary.

Corollary A.8. *Let $H \in \mathfrak{t}$ be a mass linear function on a simple 4-dimensional polytope $\Delta \subset \mathfrak{t}^*$ with no symmetric facets. Then Δ is combinatorially equivalent to a product of simplices.*

This allows us to complete the first step in classification of mass linear function on smooth 4-dimensional polytopes.

Theorem A.9. *Let $H \in \mathfrak{t}$ be a mass linear function on a smooth 4-dimensional polytope $\Delta \subset \mathfrak{t}^*$. Then there exists an inessential function $H' \in \mathfrak{t}$ so that the mass linear function $\tilde{H} = H - H'$ has the following property: at least one facet of Δ is \tilde{H} -symmetric.*

Proof. Assume that Δ has no symmetric facets. By Corollary A.8, Δ is combinatorially equivalent to a product of simplices. Moreover, by Proposition 3.25 we may assume that every facet is pervasive; hence, Δ is combinatorially equivalent to $\Delta_2 \times \Delta_2$. By Lemma 4.10, this implies that Δ is a Δ_2 bundle over Δ_2 . The result now follows immediately from Proposition 3.22. \square

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